CHAPTER 1

Complex Analysis and Surface Topology
1.1 Riemann Surfaces

1.1.1 Introduction

Topology may have had its tentative beginnings in isolated thoughts of Descartes, Leibniz, and Euler, but it was Riemann who brought the subject into the mainstream of mathematics with his inaugural dissertation in Göttingen in 1851. His introduction of the Riemann surface in that year showed the indispensable rôle of topology in questions of analysis, and thus ensured the future cultivation of the subject by the mathematical community, if only for the service of analysis. In fact, of course, Riemann surfaces were quickly seen to be of interest in themselves, and were the source of two ideas of profound significance in later topology—connectivity and covering spaces.

It hardly does Riemann justice to present only the topological aspects of his theory, however, limitations of space aside, it may be worthwhile to avoid the heavy burden of analysis found in texts on Riemann surfaces. The next section therefore presents a purely topological notion of Riemann surface, the branched covering of the sphere. Just a few words of motivation may be of value before we start.

In complex function theory it is convenient to treat the value $\infty$ as just another number, as far as possible, and one therefore completes the complex number plane by a point at infinity. The completed plane may be viewed as a sphere, since stereographic projection from the north pole $N$ of a sphere resting on the plane at the origin $O$ establishes a continuous one-to-one correspondence between the finite points $P'$ of the plane and the points $P \neq N$ on the sphere. The point $N$ itself is naturally reckoned to correspond to $\infty$ (Figure 47).

A complex function $w(z)$ can then be viewed as a map of the sphere onto itself, but of course the map need not be one-to-one, even for algebraic functions such as $z^2$. In a natural sense, $w(z) = z^2$ maps the sphere twice onto itself except at $O$ and $\infty$, since any other value of $w$ is the square of two distinct values $+\sqrt{w}$ and $-\sqrt{w}$. In fact, if we divide the $z$-sphere into hemispheres

![Figure 47](image-url)