In the historical development of linear algebra the geometry of linear transformations and the algebra of systems of linear equations played significant and important roles. A system of linear equations has the form

\[\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    &\vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n.
\end{align*}\]  

Here \(x_1, x_2, \ldots, x_n\), denote the unknowns, which are to be determined. The \(mn\) numbers \(a_{ij}, i = 1, \ldots, m, j = 1, \ldots, n\) are called the coefficients of the linear system (L) and are, of course, fixed. A solution of the system (L) is an ordered \(n\)-tuple of numbers \((s_1, \ldots, s_n)\) such that the \(m\) equations

\[\begin{align*}
    a_{11}s_1 + a_{12}s_2 + \cdots + a_{1n}s_n &= b_1 \\
    a_{21}s_1 + a_{22}s_2 + \cdots + a_{2n}s_n &= b_2 \\
    &\vdots \\
    a_{m1}s_1 + a_{m2}s_2 + \cdots + a_{mn}s_n &= b_n
\end{align*}\]

are all true. To solve the system (L) means to find all solutions of (L).

**Example 1.** Solve the linear system

\[\begin{align*}
    x_1 + x_2 + x_3 &= 3 \\
    2x_2 + x_3 &= 4 \\
    x_1 - x_2 &= -1.
\end{align*}\]
13: Systems of linear equations

**Solution.** From previous experience we recall that this system may be solved as follows:

(1) \[ x_1 + x_2 + x_3 = 3 \]
(2) \[ 2x_2 + x_3 = 4 \]
(3) \[ x_1 - x_2 = -1 \]

subtract (1) from (3) to get

\[ x_1 + x_2 + x_3 = 3 \]
\[ 2x_2 + x_3 = 4 \]
\[ -2x_2 - x_3 = -4 \]

Erase the third equation. It follows from the second, to get

\[ x_1 + x_2 + x_3 = 3 \]
\[ 2x_2 + x_3 = 4 \]
\[ x_1 = 3 - x_3 - x_2 \]
\[ 2x_2 = 4 - 2x_2 \]

or

\[ x_1 = x_2 - 1 \]
\[ x_3 = 4 - 2x_2 \]

Therefore the solutions are all triples of numbers

\( (s - 1, s, 4 - 2s) \)

where \( s \) is arbitrary. For example

\( (0, 1, 2), (-1, 0, 4) \)

are solutions but

\( (1, 1, 2) \)

is not.

On the other hand past experience should have taught us that not every system of linear equations has a solution. For example the linear system

\[ x_1 + x_2 = 1 \]
\[ x_1 + x_2 = -1 \]

clearly can have no solutions. Thus our first order of business in the study of linear equations should be to determine when a linear system has solutions, and only afterwards take up the discussion of actual techniques of solution.

Our study of matrices in the preceding chapters will come in handy here. In fact it is through the study of linear systems that matrices most frequently appear in modern scientific investigations. First let us introduce the matrices

\[ A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \]