Our objective in this section is to work out a number of numerical examples to illustrate and illuminate the theory of vector spaces we have developed so far.

Example 1. Determine whether or not the vector
\[ \mathbf{A} = (1, -2, 0, 3) \]
is a linear combination of the vectors
\[ \mathbf{B}_1 = (3, 9, -4, -2), \quad \mathbf{B}_2 = (2, 3, 0, -1), \quad \mathbf{B}_3 = (2, -1, 2, 1). \]
That is, does \( \mathbf{A} \) belong to the linear span of \( \mathbf{B}_1, \mathbf{B}_2, \) and \( \mathbf{B}_3 \), or in symbols is \( \mathbf{A} \in \mathcal{L}(\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3) \)?

Solution. Suppose that \( \mathbf{A} \in \mathcal{L}(\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3) \). Then there are numbers \( b_1, b_2, b_3 \) such that
\[ \mathbf{A} = b_1 \mathbf{B}_1 + b_2 \mathbf{B}_2 + b_3 \mathbf{B}_3. \]
Therefore
\[
\begin{align*}
(1, -2, 0, 3) &= b_1(3, 9, -4, -2) + b_2(2, 3, 0, -1) + b_3(2, -1, 2, 1) \\
&= (3b_1 + 2b_2 + 2b_3, 9b_1 + 3b_2 - b_3, \\
&\quad -4b_1 + 2b_3, -2b_1 - b_2 + b_3)
\end{align*}
\]
and therefore
\[
\begin{align*}
(1) &\quad 1 = 3b_1 + 2b_2 + 2b_3, \\
(2) &\quad -2 = 9b_1 + 3b_2 - b_3, \\
(3) &\quad 0 = -4b_1 + 0b_2 + 2b_3, \\
(4) &\quad 3 = -2b_1 - b_2 + b_3.
\end{align*}
\]
Add (4) to (2) to get

(5) \quad 1 = 7b_1 + 2b_2.

(5) yields

(6) \quad 2b_2 = 1 - 7b_1.

(3) gives

(7) \quad 2b_3 = 4b_1, \quad b_3 = 2b_1.

Putting (6), (7) into (4) gives

\[
\begin{align*}
3 &= -2b_1 + \frac{7b_1 - 1}{2} + 2b_1 \\
6 &= -4b_1 + 7b_1 - 1 + 4b_1 \\
7 &= 7b_1 \\
1 &= b_1.
\end{align*}
\]

So

\[
\begin{align*}
b_1 &= 1, \\
b_2 &= -3, \\
b_3 &= 2
\end{align*}
\]

and hence

\[A = B_1 - 3B_2 + 2B_3\]

so \(A\) is a linear combination of \(B_1, B_2, B_3\) and hence \(A\) belongs to \(\mathcal{L}(B_1, B_2, B_3)\).

**Example 2.** Determine whether or not the vector

\[A = 1 + x - 2x^2 + 4x^3\]

belongs to the subspace \(\mathcal{P}_3(\mathbb{R})\) spanned by the vectors

\[B_1 = 1 - x, \quad B_2 = 1 - x^2, \quad B_3 = 1 - x^3.\]

**Solution.** Assume that \(A\) belongs to \(\mathcal{L}(B_1, B_2, B_3)\). Then there are numbers \(b_1, b_2, b_3\) such that

\[A = b_1B_1 + b_2B_2 + b_3B_3\]

that is

\[
\begin{align*}
1 + x - 2x^2 + 4x^3 &= b_1(1 - x) + b_2(1 - x^2) + b_3(1 - x^3) \\
&= b_1 + b_2 + b_3 - b_1x - b_2x^2 - b_3x^3
\end{align*}
\]