CHAPTER IV

Vector Fields and Differential Equations

In this chapter, we collect a number of results all of which make use of the notion of differential equation and solutions of differential equations.

Let $X$ be a manifold. A vector field on $X$ assigns to each point $x$ of $X$ a tangent vector, differentially. (For the precise definition, see §2.) Given $x_0$ in $X$, it is then possible to construct a unique curve $x(t)$ starting at $x_0$ (i.e. such that $x(0) = x_0$) whose derivative at each point is the given vector. It is not always possible to make the curve depend on time $t$ from $-\infty$ to $+\infty$, although it is possible if $X$ is compact.

The structure of these curves presents a fruitful domain of investigation, from a number of points of view. For instance, one may ask for topological properties of the curves, that is those which are invariant under topological automorphisms of the manifold. (Is the curve a closed curve, is it a spiral, is it dense, etc.?) More generally, following standard procedures, one may ask for properties which are invariant under any given interesting group of automorphisms of $X$ (discrete groups, Lie groups, algebraic groups, Riemannian automorphisms, ad lib.).

We do not go into these theories, each of which proceeds according to its own flavor. We give merely the elementary facts and definitions associated with vector fields, and some simple applications of the existence theorem for their curves.

Throughout this chapter, we assume all manifolds to be Hausdorff, of class $C^p$ with $p \geq 2$ from §2 on, and $p \geq 3$ from §3 on. This latter condition insures that the tangent bundle is of class $C^{p-1}$ with $p - 1 \geq 1$ (or 2).

We shall deal with mappings of several variables, say $f(t, x, y)$, the first of which will be a real variable. We identify $D_1 f(t, x, y)$ with

$$\lim_{h \to 0} \frac{f(t + h, x, y) - f(t, x, y)}{h}.$$

§1. Existence theorem for differential equations

Let $E$ be a Banach space and $U$ an open subset of $E$. In this section we consider vector fields locally. The notion will be globalized later, and thus
for the moment, we define (the local representation of) a time-dependent 
vector field on \( U \) to be a \( C^p \)-morphism \((p \geq 0)\)

\[
f: J \times U \to E
\]

where \( J \) is an open interval containing 0 in \( \mathbb{R} \). We think of \( f \) as assigning 
to each point \( x \) in \( U \) a vector \( f(t, x) \) in \( E \), depending on time \( t \).

Let \( x_0 \) be a point of \( U \). An integral curve for \( f \) with initial condition \( x_0 \) 
is a mapping of class \( C^r \) \((r \geq 1)\)

\[
\alpha: J_0 \to U
\]
of an open subinterval of \( J \) containing 0, into \( U \), such that \( \alpha(0) = x_0 \) and 
such that

\[
\alpha'(t) = f(t, \alpha(t)).
\]

**Remark.** Let \( \alpha: J_0 \to U \) be a continuous map satisfying the condition

\[
\alpha(t) = x_0 + \int_0^t f(u, \alpha(u)) \, du.
\]

Then \( \alpha \) is differentiable, and its derivative is \( f(t, \alpha(t)) \). Hence \( \alpha \) is of class 
\( C^1 \). Furthermore, we can argue recursively, and conclude that if \( f \) is of class 
\( C^p \), then so is \( \alpha \). Conversely, if \( \alpha \) is an integral curve for \( f \) with initial condi­
tion \( x_0 \), then it obviously satisfies our integral relation.

Let

\[
f: J \times U \to E
\]

be as above, and let \( x_0 \) be a point of \( U \). By a local flow for \( f \) at \( x_0 \) we mean a mapping

\[
\alpha: J_0 \times U_0 \to U
\]

where \( J_0 \) is an open subinterval of \( J \) containing 0, and \( U_0 \) is an open subset of \( U \) containing \( x_0 \), such that for each \( x \) in \( U_0 \) the map

\[
\alpha_x(t) = \alpha(t, x)
\]
is an integral curve for \( f \) with initial condition \( x \) (i.e. such that \( \alpha(0, x) = x \)).

As a matter of notation, when we have a mapping with two arguments, 
say \( \varphi(t, x) \), then we denote the separate mappings in each argument when 
the other is kept fixed by \( \varphi_x(t) \) and \( \varphi_t(x) \). The choice of letters will always 
prevent ambiguity.