CHAPTER XVI

The Brauer-Siegel Theorem

Using the integrals expressing the zeta function, one can give certain estimates concerning its residue in order to derive asymptotic results relating the class number, regulator, and discriminant of a number field, and notably the following.

If \( k \) ranges over a sequence of number fields Galois over \( \mathbb{Q} \), of degree \( N \) and absolute value of the discriminant \( d \), such that \( N \log d \) tends to \( 0 \), then we have

\[
\log(hR) \sim \log d^{1/2}.
\]

One may of course ask whether it is possible to lift the restriction of normality, and the condition that \( N \log d \) tends to \( 0 \). With the present approach, these questions involve Artin's conjecture on the non-abelian \( L \)-series and the Riemann hypothesis (as will be clear in the proof). The existence of infinite unramified extensions proved by Golod-Shafarevic shows that the assumption \( N \log d \rightarrow 0 \) is necessary. Indeed, if \( k \) is a number field admitting an infinite tower of unramified extensions \( K \), then \( N_k/\log d_k \) is constant.

We observe that the discriminant of the field \( k = \mathbb{Q}(\zeta) \), where \( \zeta \) is a \( p \)-th root of unity (\( p \) a prime), is \( d_k = p^{p-2} \) and so our statement applies to such fields. Similarly for towers of \( p^\ell \)-th roots of unity.

The study of the behavior of \( N/\log d \) is thus of considerable interest. We shall use the essentially elementary fact that for all number fields with

\[
N > 1 \quad \text{(i.e. } k \neq \mathbb{Q})
\]

the number \( N/\log d \) is bounded. This follows at once from Minkowski’s theorem that in every ideal class there exists an integral ideal \( \alpha \) such that

\[
\text{Na} \leq C_k d^{1/2},
\]

where \( C_k \) is the Minkowski constant. Taking the \( N \)-th root, a simple computation, using the fact that \( 1 \leq \text{Na} \), shows that there is an absolute constant \( C \) such that \( N/\log d \leq C \).
§1. An upper estimate for the residue

**Lemma 1.** There exists an absolute constant \( c_1 \) such that the inequality

\[
\kappa(k) \leq c_1^N(1 + \alpha)^N d_k^{1/2\alpha} \quad (N = [k : \mathbb{Q}])
\]

holds for all number fields \( k \) and all \( \alpha \geq 1 \).

*Proof.* According to Chapter XIV, Theorem 14, Corollary 3 and Theorem 15 together with the fact that the integrals expressing the zeta function are \( \sim \) for real \( s \), we get for \( s > 1 \):

\[
(2^{-2r_2\pi^{-N}d_k})^{s/2} \Gamma^{r_1}(\frac{s}{2}) \Gamma^{r_2}(s) \xi_k(s) \geq \kappa \frac{d_k^{1/2}(2\pi)^{-r_2}}{s(s - 1)}.
\]

If we put \( s = 1 + \alpha^{-1} \), then the gamma factors are uniformly bounded. We have obvious contributions of type \( c_1^N \) and \( d_k^{1/2\alpha} \). From the product expansion for the zeta function we have the inequalities

\[
\xi_k \left( 1 + \frac{1}{\alpha} \right) \leq \xi_\mathbb{Q} \left( 1 + \frac{1}{\alpha} \right)^N \leq (1 + \alpha)^N.
\]

The lemma follows at once.

**Lemma 2.** There exists a constant \( c_2 \) such that for \( k \neq \mathbb{Q} \),

\[
\log(hR)/\log(d^{1/2}) \leq c_2.
\]

If \( k \) ranges over a sequence of fields such that \( N/\log d \) tends to 0, then for this sequence

\[
\lim \sup \left[ \left( \frac{\log hR}{\log d^{1/2}} - 1 \right) \frac{1}{N} \right] \leq 0.
\]

*Proof.* We use the elementary estimate that the number of roots of unity \( w \) in a number field \( k \) is \( \leq c_3 N^2 \) for some absolute constant \( c_3 \). (Use the fact that the field of \( n \)-th roots of unity over \( \mathbb{Q} \) has degree \( \varphi(n) \), together with an obvious estimate of \( \varphi(n) \), using \( \varphi(p^r) = (p - 1)p^{r-1} \) and the multiplicativity.)

From Lemma 1, and the value for \( \kappa \), we get

\[
\frac{\log hR}{\log d^{1/2}} - 1 \leq \frac{N}{\log d^{1/2}} \log(c_1(1 + \alpha)) + \frac{1}{\alpha} + \frac{N}{\log d^{1/2}} \log c_2.
\]

Putting \( \alpha = 1 \) proves the first assertion. Fixing \( \alpha \), and taking our sequence of fields shows that for each \( \alpha \) large, and all but a finite number of fields in our sequence, the difference on the left is \( \leq \alpha^{-1} + \epsilon \) with arbitrarily small \( \epsilon \). This proves our assertion.