2 The Electromagnetic Field of a Known Charge Distribution

2.1 The Stationary-Action Principle and Conservation Theorems

If the field equations originate from a stationary-action principle, then a conserved current can be constructed for each parameter of an invariance group.

Field theory may be regarded as a generalization of the mechanics of point particles, in which the dynamical variables $q_i(t)$ are replaced with fields $\Phi(x, t)$, such as $E(x, t)$ and $B(x, t)$. The discrete index $i$ goes over to the continuous variable $x$, and, accordingly, the sum $\sum_i$ is replaced with an integral $\int d^3x$. A direct transcription of the formalism of I, §3, leads to infinite-dimensional manifolds, which we would prefer to avoid. Instead, we merely generalize the stationary-action principle (1.2.3.20) in order to find the analogues of the constants arising from the invariance properties. It is clear that in field theory the action $\int dt L(q, \dot{q})$ involves an integral over a four-dimensional submanifold $N_4$, and thus requires a 4-form, which allows the construction of a chart-independent integral.

The Lagrangian Formulation of Field Theory (2.1.1)

The action is given by

$$W = \int_{N_4} \mathcal{L}(\Phi, d\Phi),$$
where $\mathcal{L} \in E_4$ is the **Lagrangian**. The field equations result from the requirement that $\delta W = 0$ for $N_4$ compact and $\forall \Phi$ such that $\delta \Phi \mid_{\partial N_4} = 0$.†

If we strengthen the homogeneous Maxwell equations to $F = dA$, then in pseudo-Riemannian space, the appropriate

**Electromagnetic Lagrangian** (2.1.2)

is

$$\mathcal{L} = -\frac{1}{2} dA \wedge *dA - A \wedge *J.$$  

**Proof**

Making a variation $A \rightarrow A + \delta A$ and using (1.2.18) (a), one finds

$$-\delta W = \int_{N_4} \delta A \wedge [*J + d^*dA] + \int_{\partial N_4} \delta A \wedge *dA,$$

which vanishes if $\delta A \mid_{\partial N_4} = 0$ and $d^*F = -*J$. □

**Remarks** (2.1.3)

1. The variational formulation offers no guarantee of existence or uniqueness of the solutions of the field equations. Nowhere has it been assumed that $d^*J = 0$, though without this condition it is not possible to satisfy $\delta W = 0$ for $\forall \delta A$ such that $\delta A \mid_{\partial N_4} = 0$. The reason is easy to discover. With the gauge transformation $A \rightarrow A + d\Lambda$, where $\Lambda \mid_{\partial N_4} = 0$, $W$ changes by $\int_{N_4} \Lambda d^*J$, and is linear in $\Lambda$ not only for infinitesimal $\Lambda$. As a linear functional, either $W$ has no stationary points, or else, if $d^*J = 0$, it has a plateau. Accordingly, either there are no solutions at all, or else the solution is not uniquely fixed by any boundary condition whatsoever, because there is always the possibility of a gauge transformation.

2. According to (1: 5.2.8),

$$-\frac{1}{2} F \wedge *F = -\frac{1}{2} F_{\sigma \rho} F^\sigma \rho *1 = \frac{1}{2}(|E|^2 - |B|^2)*1.$$  

The sign of $\mathcal{L}$ has been chosen so that the interaction

$$-A \wedge *J = -*_{i_j} A = -J^x A_x *1$$

of a point particle moving along the world-line $z(s)$ (cf. (1.3.25; 2)) has the same sign

$$-e \int_{-\infty}^{\infty} ds \: \dot{z}^x(s) A_x(z(s)) \delta^4(x - z(s)) *1$$

† We use the symbol $\delta$ in § 2.1 for variations, rather than for codifferentials.