8. FOURIER SERIES AND INTEGRALS (FUNDAMENTAL PRINCIPLES)

The following fundamentals and elementary facts are standard mathematical knowledge today, and can be found in a great number of textbooks in analysis. As a general reference, we mention [Dym-McKean 1972].

A. Fourier Series

\[ C^0(S^1) \]
Banach space of continuous (complex-valued) functions on the circle \( S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \) with the norm \( ||f||_\infty := \sup \{|f(z)| : |z| = 1\} \) for \( f \in C^0(S^1) \).

\[ L^1(S^1) \]
Banach space of (complex-valued) integrable functions on \( S^1 \) with the norm \( ||f||_1 := \int_{S^1}|f| \). Often one represents \( L^1(S^1) \) via the space \( L^1[0,1] \) of integrable functions on the interval \([0,1]\) and then has \( ||f||_1 = \int_0^1 |f(x)| \, dx \).

\[ L^2(S^1) \]
Hilbert space of square-integrable functions on \( S^1 \) with the inner product \( \langle f, g \rangle := \int_{S^1} f(x) \overline{g(x)} \, dx \) and the associated norm \( ||f|| := \left( \int_0^1 |f(x)|^2 \, dx \right)^{1/2} \). Here we have identified the space \( L^2(S^1) \) with \( L^2[0,1] \).

**Warning:** Functions in \( L^1(S^1) \) or \( L^2(S^1) \) are identified if they agree outside a set of measure zero. In particular, \( f \) is identified with the zero function, if \( f \) is zero "almost everywhere"; i.e., \( f \) is non-zero on at most a set of measure zero. In this way, we have \( ||f|| = 0 \) precisely when \( f = 0 \).

Thus, strictly speaking, the elements of \( L^1(S^1) \) or \( L^2(S^1) \) are not functions, but rather equivalence classes of functions. While this is true, in practice it is much simpler and generally harmless to disregard this fine distinction, and we will do this in what follows.

**Exercise 1.** Show that \( L^2(S^1) \) with \( \langle \cdot, \cdot \rangle \) is indeed a Hilbert space. We need to show:

a) \( \langle \cdot, \cdot \rangle : L^2(S^1) \times L^2(S^1) \to \mathbb{C} \) is well defined. **Hint:** The pointwise estimate
\[
2|f(z)g(z)| \leq |f(z)|^2 + |g(z)|^2
\]
shows that \( fg \in L^1(S^1) \) for \( f, g \in L^2(S^1) \).

b) Symmetry (i.e., \( \langle f, g \rangle = \langle g, f \rangle \)), bilinearity (linearity in the first slot, and conjugate linearity in the second), and positivity (i.e.,
\[<f,f> \geq 0 \text{ and } <f,f> = 0 \text{ exactly when } f = 0 \] hold. (This is trivial.)

c) \(L^2(S^1)\) is a complex vector space. **Hint:** For closure under addition, prove Hermann Minkowski's inequality

\[
(f \ast g)^2 \leq (f^2)^{1/2} + (g^2)^{1/2}
\]

(= Triangle Inequality).

d) \(L^2(S^1)\) is complete. In order to prove that a Cauchy sequence \(\{f_n\}_{n=1}^{\infty}\) with \(f_n \in L^2(S^1)\) and \(\|f_n - f_m\|_2 \to 0\) possesses a limit \(f \in L^2(S^1)\) with \(\|f - f_n\| \to 0\), one must apply the fundamental convergence theorems which distinguish the Lebesgue integral from the Riemann integral. The rather technical proof can be found in [Dym-McKean, 16-20].

e) \(L^2(S^1)\) is separable. **Hint:** Show that the family of piecewise constant functions, having rational real and imaginary parts and with jumps at finitely many rational points, is dense in \(L^2(S^1)\).

**Supplement:** With the help of the smoothing functions of the kind

\[
g(x) = \begin{cases} 
0 & \text{for } x \leq 0 \\
\exp \left( \frac{1}{x}(1/x-1) \right) & \text{for } 0 < x < 1 \\
0 & \text{for } 1 \leq x,
\end{cases}
\]

it follows that even \(C^\infty(S^1)\) is dense in \(L^2(S^1)\).

**Convolution:** In \(L^1(S^1)\), there is a commutative and associative product ('"convolution") given by

\[
(f \ast g)(x) = \int_{-1}^{1} f(x-y)g(y)dy,
\]

which makes \(L^1(S^1)\) an algebra (without identity). In particular, we have

\[
\|f \ast g\|_1 \leq \|f\|_1 \cdot \|g\|_1.
\]

Moreover, one can show that, relative to \(\ast\), \(L^2(S^1)\) is an ideal in \(L^1(S^1)\); hence, \(f \ast g\) is in \(L^2(S^1)\), whenever one of the factors lies in \(L^2(S^1)\). See [Dym-McKean, 41].

**Orthonormal systems:** The family \(\{z^n : n \in \mathbb{Z}\}\), where \(z^n : S^1 \to \mathbb{C}\) is the function that assigns to each \(z \in S^1\) the value \(z^n\), is an orthonormal system in \(L^2(S^1)\). Regarding \(L^2(S^1)\) as \(L^2[0,1]\), the