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Mathematical Models and Their Formulation

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19.1 MATHEMATICAL MODELING

19.1.1 The Continuum Mechanics Model of Matter

With a few exceptions, the chapters of this handbook are concerned with mathematical methods useful in the quantitative analysis of problems in science and engineering. An important and challenging aspect of any quantitative study of a real-life phenomenon is the formulation of mathematical problems which are relevant to a better understanding of the phenomenon and to which these mathematical methods can be applied. Real-life phenomena are usually too complex to be analyzed quantitatively without idealization and simplification. It is just not feasible or practical to follow the individual motions of trillions of molecules in a cubic centimeter of air or the evolution of billions of stars in a typical galaxy. For many practical purposes, however, information about a body of matter (or a galaxy) can be obtained by treating the collection of molecules (or stars) in that body as a “continuum medium” having properties, such as density, velocity, etc., that vary smoothly throughout the body. In this continuum model, the equilibrium or motion of the body under external forces and torques, for example, may be taken as a consequence of Euler’s laws of mechanics for continuous media. The mathematical methods described in this handbook may now be used to deduce from Euler’s law an initial/boundary-value problem for differential equations that governs the mechanical behavior of the continuum. Section 4.7 of this handbook gives a sample derivation of some relevant differential equations of this mathematical model, widely known as continuum mechanics, for the study of the mechanics of deformable bodies of matter.

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†Here, science and engineering are to be taken in the most general context. For example, we would include in science, the quantitative aspects of archaeology, commerce, etc.
For more than two centuries, countless macroscopic phenomena involving mechanical behavior of matter (solid, fluid, or gas) have been explained, and the effects of any change in their setting have been accurately assessed by the continuum mechanics model, with the help of mathematical methods described in this handbook and elsewhere. This article is not concerned with the contents of the continuum mechanics model; nor is a knowledge of the model a prerequisite for the rest of the chapter. (The construction of the mathematical models of this chapter will start from first principles.) We merely wish to identify the principles operating in the development of this extremely successful mathematical model to be used in the construction of new mathematical models of current interest.

Beyond the process of idealization and application of the laws of mechanics already discussed above, another important aspect in the model development is the process of simplification. In actual applications of the continuum mechanics model, it is usually very difficult or costly to extract useful information about the phenomena from this sophisticated mathematical model without some simplifications. Beyond those resulting naturally from a certain uniformity or symmetry inherent in a particular problem, simplifications of the model through judicious approximations are often made for special classes of problems, based on the analyst's understanding of the phenomenon, retaining only those features in the model expected to be most relevant. Beam, plate, and shell theories in solid mechanics and water waves (see section 15.2.6) and lubrication theory in fluid mechanics are examples of such simplifications for different classes of problems to which the general continuum mechanics model applies. For specific problems where one of these simplified theories is appropriate, further idealization and approximation may still be possible (and sometimes necessary!) to make progress toward its solution. We saw in section 15.2.6 the examples of small amplitude approximation, shallow water theory, and far-field expansion (leading to the Korteweg-deVries equation) for water waves. At other times, simplification, idealization, and approximation with very little mathematical or scientific justification may have to be made to bring the continuum mechanics model to bear on a problem. A real-life example of this situation will be described in section 19.6. The discussion there illustrates why applied mathematicians must have a good command of the scientific and engineering aspects of the phenomenon under investigation if they are to be successful in mathematical modeling. It is simply not enough to be adroit in mathematical methods alone.

19.1.2 Construction of Mathematical Models

The various steps in the development of the continuum mechanics model of matter, briefly described in section 19.1.1, are basic to constructing mathematical models for any real-life situations. The principal steps consist of

1. the idealization of the actual phenomenon of interest (such as space-time continuum mentioned in section 19.1.1)
2. the introduction of mathematical quantities (such as the density and velocity fields for continuous media) for an adequate description of the behavior of the idealized phenomenon