INTERPOLATION OPERATORS AS OPTIMAL RECOVERY SCHEMES FOR CLASSES OF ANALYTIC FUNCTIONS

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1. INTRODUCTION

Suppose that the only information we have about a function \( f \) is that it belongs to a certain class \( \mathcal{B} \) (usually a ball in a normed space) and that it takes on given values at some finitely many points \( x_1, \ldots, x_n \) of its domain (or that some other finitely many linear functionals \( \ell_1, \ldots, \ell_n \) have given values at \( f \)). Suppose we are to assign a value to \( f(x) \) where \( x \) does not belong to the set \( \{x_1, \ldots, x_n\} \) (or to \( \ell(f) \) where \( \ell \) is not in the span of \( \{\ell_1, \ldots, \ell_n\} \)). The value \( a^* \) for \( f(x) \) is considered optimal if for any other assignment \( a \) there is some \( f_a \in \mathcal{B} \) with \( f_a(x_i) = f(x_i) \) (\( i = 1, \ldots, n \)) for which the error \( |a - f_a(x)| \) is at least as large as \( |a^* - f(x)| \). If moreover a function \( s_* \in \mathcal{B} \) can be found such that \( s_* \) evaluated at \( x \) gives the optimal value \( a^* \) and this is so for every \( x \) in the domain of the functions \( f \) then \( s_* \) is considered an optimal interpolant (or extrapolant) for these functions.

If \( \mathcal{B} \) is a closed ball in a Hilbert space (and in some other cases) it is known that an optimal interpolant \( s_* \) exists (see [1], [2]). The interpolant \( s_* \) has several other extremal properties, and is in particular characterized uniquely by the property that among all functions in \( \mathcal{B} \) which have the given values at \( x_1, \ldots, x_n \), \( s_* \) has minimum norm. This allows \( s_* \) to be calculated from the variational equation and the interpolation conditions. Most of these interpolants are called splines of one kind or another.

More recently the problem of reconstructing in an optimal way the functions \( f \) of a class \( \mathcal{B} \) from data \( f(x_1), \ldots, f(x_n) \) has
been given a wider formulation. If $F_n$ denotes the data vector \( \{f(x_1), \ldots, f(x_n)\} \) one asks for a mapping $S$ (linear or non-linear) from the space of data vectors to the class $\mathcal{D}$, so that $SF_n$ is a good substitute for the only partially known function $f$. Each such mapping is called a recovery scheme and $S_*$ is said to be an optimal recovery scheme (with respect to a chosen norm or seminorm) if the norm of the largest error $SF_n - f$ that occurs in $\mathcal{D}$ is minimized by $S_*$ (for this concept and results based on it see [3], [4]).

It is not too surprising that in the case where $\mathcal{D}$ is a ball in a Hilbert space the two problems lead to the same solution: $S_* F_n = s_*$. This is true, no matter what the norm or seminorm is that is used for measuring the error $Ef = SF_n - f$. Those used most commonly in this paper are $|Ef(x_0)|$ for some fixed $x_0$ in the domain of $f \in \mathcal{D}$, and $\sup_{x \in D} |Ef(x)|$ for some subset $D$ of the domain. The limitation imposed by the Hilbert space setting is compensated for by the simplicity of the recovery algorithm. The mapping $S_* : F_n \mapsto s_*$ is, of course, linear, moreover the mapping $f \mapsto s_*$ is an orthogonal projection onto the $n$-dimensional subspace of spline interpolants with nodes at $x_1, \ldots, x_n$. These and other general results, which are hardly new, are presented in Sections 2-3 below. Practically all the Hilbert spaces of functions of use for numerical analysis have a reproducing kernel. This must be so because the place value functional $x \mapsto f(x)$ is necessarily continuous. This is, in particular, true of the spaces of analytic functions in one complex variable that are treated in detail in this paper. In Section 4 formulas for the optimal recovery operator and the optimal error in terms of the reproducing kernel are presented.

The "splines" of our Hilbert spaces of analytic functions are themselves analytic functions. The higher order discontinuities - the "knots" of the usual polynomial splines - are replaced by poles outside of the domain of analyticity, which have a simple geometric relationship to the interpolation nodes. The spaces considered in Sections 5-8 have been selected so that they yield simple spline bases, which can be obtained without computation. This is not to imply that the methods are restricted to these examples. Simplicity is achieved by considering only simple domains (disk, annulus, strip, ellipse) and specially chosen norms. These norms, although not the most natural (or traditional) ones from the point of view of function theory, are topologically equivalent to them, and the metric difference is rather irrelevant to the numerical analyst. If, for example, in the case of the strip (with periodicity) the area integral of the square of the absolute value had been chosen for the norm, as was done in some recent papers for similar problems (see [5], [6]), the resulting splines would be elliptic functions rather than the elementary ones obtained by our choice of norm (this will be shown in a forthcoming note).