GENERAL MANY-BODY SYSTEMS

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INTRODUCTION

The problem of how to visualize and sometimes solve a general many-body system is considered. The ideas are established in the context of very simple small systems, a Hubbard model and a coupled electron-phonon model. These models are also solved to good approximation in the thermodynamic limit, although the Hubbard model is restricted to a small number of holes added to the Mott insulating state. Response functions are also considered.

A fairly general many-body Hamiltonian is

\[ H = H_{el} + H_{el-el} + H_{el-ph} + H_{ph}, \]

consisting of an electron or other fermion kinetic energy and electron-electron interactions, which may be coupled to a bose field such as a phonon. The phonons themselves may be nonlinear (have self-interactions). The system may be strongly coupled (\( H_{el-el} \) and \( H_{el-ph} \) may be large). One may also add coupling to an external driving field, such as an AC electric field. The methods discussed are nonperturbative, and so differ from the standard methods of diagrammatic perturbation theory. A comparison is made with diagrammatic methods in the context of the random phase approximation.

HUBBARD MODEL (SMALL SYSTEM)

The first example is the Hubbard model, which describes interacting electrons and contains only the first two terms of Eq. (1):

\[ H_{II} = -t \sum_{<j,k>,s} (c_j^+ c_k + h.c.) + U \sum_j c_j^+ c_j c_j^+ c_j. \]

The operator \( c_j^+ \) creates an electron of spin \( s \) on a Wannier orbital on lattice site \( j \). The first term (electron kinetic energy) causes electrons to hop to nearest neighbor sites without changing their spin. The last term is a repulsive on site electron-electron interaction. To illustrate the exact solution of Eq. (2) for a small system, consider the problem with two sites, two electrons, and the \( z \)-component of the spin \( S_z \), which is conserved, equal to zero. (Infinite systems will be considered later.)

The Hilbert space in coordinate representation is given by
\begin{align}
|1\rangle &= \uparrow \downarrow \\
|2\rangle &= \uparrow\downarrow \\
|3\rangle &= \uparrow\downarrow \\
|4\rangle &= \downarrow \uparrow, 
\end{align}

where the first site is on the left and the second on the right. The Hamiltonian operating on $|1\rangle$ connects (has nonzero matrix elements) to states $|2\rangle$ and $|3\rangle$. State $|4\rangle$ also connects to $|2\rangle$ and $|3\rangle$, see Fig. (1). The diagonal energy of states $|2\rangle$ and $|3\rangle$ is $U$, and that of $|1\rangle$ and $|4\rangle$ is $0$.

Note that an interacting many-body problem (containing the product of four fermion operators) has been mapped onto a non-interacting one particle tight-binding problem. If the operator $b_j^\dagger$ creates many-body state $|j\rangle$ in Eq. (3), the new Hamiltonian is

$$
\hat{H} = \sum_{j,k} \tilde{t}_{jk} (b_j^\dagger b_k + \text{h.c.}) + \sum_j \varepsilon_j b_j^\dagger b_j,
$$

with no interactions (four-fermion operators). The sites in the tight-binding problem, however,

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{tight_binding.png}
\caption{The tight-binding model represents the two-site Hubbard model with two electrons, $S_z=0$. The bonds are off-diagonal matrix elements of amplitude $-t$.}
\end{figure}

represent many-body states, not the usual atomic or Wannier orbitals. It is a general result that one can always exactly map the ground and excited states of an interacting many-body problem onto those of a noninteracting one-body problem in this way.

In matrix notation, $\hat{H}$ is

$$
\hat{H} = \begin{bmatrix}
0 & -t & -t & 0 \\
-t & U & 0 & -t \\
-t & 0 & U & -t \\
0 & -t & -t & 0
\end{bmatrix}.
$$

This matrix is simple to diagonalize exactly. The four eigenvalues are

$$(E_1,E_2,E_3,E_4) = (\frac{U - \sqrt{U^2 + 16t^2}}{2}, 0, U, \frac{U + \sqrt{U^2 + 16t^2}}{2}).$$

The two lowest energy (unnormalized) eigenvectors are

$$
|\psi_1\rangle = |1\rangle + |4\rangle + a (|2\rangle + |3\rangle)
$$

$$
|\psi_2\rangle = |1\rangle - |4\rangle,
$$

where $a = -E_1 / 2t$. $|\psi_1\rangle$ is a singlet state and $|\psi_2\rangle$ is the $S_z=0$ triplet state. For $U \gg t$, the low energy part of the Hilbert space is described by

$$
H = \text{const} + J \hat{\sigma}_1 \cdot \hat{\sigma}_2,
$$

with $J = t^2/U$. 

254