SECTION XV
Property $S$ and Uniformly Locally Connected Sets

**Definition.** A set $M$ in a metric space is said to have *property $S$* if and only if for every $\varepsilon > 0$, $M$ is the union of a finite number of connected sets, each of diameter less than $\varepsilon$.

**Definition.** A set $M$, also in a metric space, is said to be *uniformly locally connected* if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that any pair of points $x, y$ of $M$ with $\rho(x, y) < \delta$ lie together in a connected subset of $M$ of diameter less than $\varepsilon$.

**EXERCISES XV.**

1. Prove that if $M$ has property $S$, so also has each set $M_0$ satisfying the relation $M \subset M_0 \subset \text{Cl}(M)$.

   For the following exercises, consider the diagram:

   ![Diagram](local_connectedness_property_S)

   2. Prove the indicated implications.

   3. Decide which of the implications are reversible.

   Validate your conclusions with proofs or counterexamples.
1. Prove that if \( M \) has property \( S \), so also has each set \( M_0 \) satisfying the relation \( M \subset M_0 \subset \text{Cl}(M) \).

We first note that \( \text{diam } M = \text{diam } \text{Cl}(M) \). Since \( M \subset \text{Cl}(M) \), \( \text{diam } M \leq \text{diam } \text{Cl}(M) \). Let \( \varepsilon > 0 \) be given, and let \( x, y \in \text{Cl}(M) \). Then there exist points \( x', y' \in M \) such that \( \rho(x,x') < \varepsilon/2 \) and \( \rho(y,y') < \varepsilon/2 \). Then

\[
\rho(x,y) \leq \rho(x,x') + \rho(x',y') + \rho(y',y) \leq \varepsilon + \text{diam } M
\]

Since \( x \) and \( y \) were arbitrary, as was \( \varepsilon \), we have \( \text{diam } \text{Cl}(M) \leq \text{diam } M \).

Now suppose \( \varepsilon > 0 \) is given and \( M = \bigcup M_i \), where each \( M_i \) is connected and has diameter less than \( \varepsilon \). Then \( M_0 = \bigcup \text{Cl}(M_i) \cap M_0 \), where \( \text{Cl}(M_i) \cap M_0 \) is connected for each \( i \) (since \( M_i \subset \text{Cl}(M_i) \cap M_0 \subset \text{Cl}(M_i) \)) and \( \text{diam } \text{Cl}(M_i) \cap M_0 \) is less than \( \varepsilon \). Hence \( M_0 \) has property \( S \).

2. Prove the indicated implications.

(a) Property \( S \Rightarrow \) local connectedness. Let \( M \) be a set in a metric space with property \( S \); let \( \varepsilon > 0 \) be given, and choose \( p \in M \) with \( \varepsilon \)-sphere centered at \( p \). Then \( M = \bigcup M_i \), where each \( M_i \) is connected and has diameter less than \( \varepsilon/2 \). If \( K \) is the union of all \( M_i \)’s satisfying \( p \in \text{Cl}(M) \), then \( K \) is connected. Also, if \( x, y \in K \), then \( \rho(x,y) \leq \rho(x,p) + \rho(p,y) \), and since \( x \in M_i \), \( y \in M_j \), and \( p \in \text{Cl}(M_i) \cap \text{Cl}(M_j) \) for some \( i \) and \( j \), we have \( \rho(x,y) < \varepsilon/2 + \varepsilon/2 = \varepsilon \). Thus the diameter of \( K \) is less than \( \varepsilon \). Since \( p \) cannot be a limit point of \( K \), there is an open set \( V \) containing \( p \) and completely contained in \( K \). Then the open set \( U \setminus V \) satisfies \( p \in U \setminus V = U \) and \( (U \setminus V) \cap M = K \), where \( K \) is a connected subset of \( U \). Hence \( M \) is locally connected.

(b) Local connectedness, compactness \( \Rightarrow \) uniform local connectedness. Let \( M \) be compact and locally connected in a metric space, and let \( \varepsilon > 0 \) be given. We can cover \( M \) with regions in \( M \) of diameter less than \( \varepsilon/2 \), so that using the compactness of \( M \), we have \( M = \bigcup R_j \), where each \( R_j \) is open in \( M \), is connected, and has diameter less than \( \varepsilon/2 \). For each pair \( R_i, R_j \) satisfying \( \text{Cl}(R_i) \cap \text{Cl}(R_j) \neq \emptyset \), let \( \delta \geq \rho(R_i,R_j) \), so that each \( \delta \geq 0 \). Define \( \delta \) by \( \delta = \min \{ \delta_{ij} \} \) if \( \{ \delta_{ij} \} \neq \emptyset \); otherwise, \( \delta = \text{diam } M \). Now choose \( x, y \in M \) such that \( \rho(x,y) < \delta \). Then for some \( i \) and \( j \), we have \( x \in \text{Cl}(R_i) \), \( y \in \text{Cl}(R_j) \), where \( R_i \cap R_j \neq \emptyset \). The set \( \text{Cl}(R_i) \cup \text{Cl}(R_j) \) then has the properties of being connected, containing \( x \) and \( y \), and having diameter less than \( \varepsilon \). Hence \( M \) is uniformly locally connected.

(c) Uniform local connectedness, conditional compactness \( \Rightarrow \) property \( S \). Let \( M \) be a uniformly locally connected, conditionally compact set in a metric space, and let \( \varepsilon > 0 \) be given. Then there exists a \( \delta > 0 \) such that if \( \rho(x,y) < \delta \), then \( x \) and \( y \) lie in a connected subset of diameter less than \( \varepsilon/2 \). Since \( \text{Cl}(M) \) must be compact, we know that \( \text{Cl}(M) \) is separable, and hence so also is \( M \). Let \( P = \{ p_i \} \) be a countable dense set in \( M \). Define, for each \( n \), the set \( R_n \) to be the set of all those points of \( M \) which lie together with \( p_n \) in a connected set of diameter less than \( \varepsilon/2 \). Then \( R_n \) is connected and has diameter less than \( \varepsilon \). We claim that \( M = \bigcup R_n \) for some \( N \). Suppose not; then there exists an infinite subsequence \( \{ p_{n_i} \} \) of \( \{ p_i \} \) such that for each \( i \), \( p_{n_i} \) is not contained in \( \bigcup_{i=1}^{n_i-1} R_n \). Since \( M \) is conditionally compact, \( \{ p_{n_i} \} \) has a limit point \( p \). Then for some \( n_i \) and \( n_j \) \( (i > j) \), \( \rho(p_{n_i}, p_{n_j}) < \delta \), so that \( p_{n_j} \in R_{n_i} \subset \bigcup_{i=1}^{n_i-1} R_n \). But this contradicts the construction of \( \{ p_{n_i} \} \). Hence \( M = \bigcup R_i \) for some \( N \), and \( M \) has property \( S \).