NOTES ON THE INHOMOGENEOUS SCHRÖDINGER EQUATION

by

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In [1] and [2] we discussed the solution of the homogeneous Schrödinger equation $(\frac{\Delta}{2} + q)u = 0$ with boundary condition. It is customary in classical analysis to treat this problem as equivalent to the solution of the corresponding inhomogeneous equation $(\frac{\Delta}{2} + q)u = \phi$ with vanishing boundary condition, by a simple substitution. However, sufficient smoothness of the given data is required for this method. It turns out that the probabilistic approach is easily adapted to the inhomogeneous case, via the potentials. Relatively mild assumptions are sufficient for the purpose. Whereas it is possible to treat the problem in a "purely analytic" setting based on old and new Green's functions, we follow a different route and carry out the calculations by integrations over time rather than over space.

Let $D$ be a domain in $\mathbb{R}^d$, $d \geq 1$, with $m(D) < \infty$, where $m$ is the Lebesgue measure in $\mathbb{R}^d$. No regularity assumption is imposed on $\partial D$. Define a class of functions, to be denoted by $L^*(D)$, as follows:

$\phi \in L^*(D)$ iff $\phi$ is locally bounded in $D$ and $\phi \in L^1(D, m)$. Then $L^*(D)$ is a linear space which admits the operation $\phi \mapsto |\phi|$, and multiplication by a bounded measurable function.

Let $q$ be a bounded Borel measurable function on $\mathbb{R}^d$, $Q = \sup_{x \in \mathbb{R}^d} |q(x)|$; *Research supported in part by NSF grant MCS83-01072 at Stanford University.
\{X_t, \ t \geq 0\} \text{ the standard Brownian motion in } \mathbb{R}^d; \\
\ e_q(t) = \exp\left[\int_0^t q(X_s)ds\right]; \\
and \ \tau = \tau_D \text{ the first exit time from } D. \text{ Define a semigroup } \{L^t_q, \ t \geq 0\} \\
as follows: \text{ for positive Borel measurable } f, \\
L^t_q f(x) = \mathbb{E}^x\{t < \tau_D; e_q(t)f(X_t)\}.

The associated potential will be denoted by \( V(q) \):

\[ V(q)f = \int_0^\infty L^t_q f \, dt. \]

These notations are the same as in [2], except for the explicit indication of \( q \). For \( q = 0 \), \( \{L^t_0\} \) reduces to the semigroup of the Brownian motion killed outside \( D \); and \( V(0) \) becomes the classical Green's potential for \( D \).

The gauge for \( (D,q) \) is defined in [1] to be the function \( \mathbb{E}^x\{e_q(\tau_D)\} \) for \( x \in D \). It is proved that (Theorem 3.1 of [1]) the gauge is bounded in \( \overline{D} \) if and only if \( V(q)_1 < \infty \) in \( D \). We shall assume this condition throughout this note.

**PROPOSITION 1.** \( V(q) \) maps \( L^\infty(D) \) into \( L^\infty(D) \).

**PROOF:** If \( \phi \in L^1(D,m) \), then by (4) and (7) of [2],

\[ \int_1^\infty L^t_q \phi \, dt \text{ is bounded in } D. \]

If \( \phi \in L^\infty(D) \), then we have