Congruent Embedding into Boolean Vector Spaces

N. V. SUBRAHMANYAM
ANDHRA UNIVERSITY
Waitair, India

1. INTRODUCTION

Let $B = (B, +, \cdot, ')$ be a Boolean algebra, whose null and universal elements will be denoted by 0 and 1, respectively. By a vector space over $B$ (or simply, a $B$-vector space) is meant an additive group $V$ (whose "zero" also will be denoted by 0), together with a mapping: $(a, x) \rightarrow ax$ of $B \times V$ into $V$ such that (a) $a(x + y) = ax + ay$, (b) $(ab)x = a(bx)$, (c) $1x = x$ and (d) $(a + b)x = ax + bx$ whenever $ab = 0$. Further, a $B$-vector space $V$ is said to be normed if and only if to each $x \in V$ is associated a unique element $|x| \in B$ such that (a) $|x| = 0$ if and only if $x = 0$ and (b) $|ax| = a|x|$ for all $x \in V$ and $a \in B$. If $V$ is a normed $B$-vector space and we put $d(x, y) = |x - y|$, then $(V, d)$ is a $B$-metric space in the sense of the following:

Definition: By a $B$-metric space $S$ is meant a pair $(S, d)$, where $S$ is a set and $d: S \times S \rightarrow B$ is a mapping such that (a) $d(x, y) = 0$ if and only if $x = y$ and (b) $d(x, z) \leq d(x, y) + d(z, y)$ for all $x, y, z$ in $S$.

By a basis of a $B$-vector space $V$ is meant a nonempty subset $G$ of $V$ such that (a) if $a_1, a_2, \ldots, a_n \in B, g_1, g_2, \ldots, g_n \in G, a_1a_2 \ldots a_n \neq 0$ and $a_1g_1 + a_2g_2 + \ldots + a_ng_n = 0$, then $g_1 + g_2 + \ldots + g_n = 0$, and (b) if $x \in V$, then there exist $g_1(x), g_2(x), \ldots, g_n(x) \in G$ and $a_1(x), a_2(x), \ldots, a_n(x) \in B$ such that $a_i(x) \cdot a_j(x) = 0$ for $i \neq j$ and
Every $B$-vector space $V$ with a basis is necessarily normed, although not every normed $B$-vector space has a basis (see theorem 8, Ref. 4, p. 429). If $G$ is a basis of a $B$-vector space $V$ and $x \in V$, then $|x - g| = 1$ except for a finite number of $g$'s in $G$ and $x$ has a unique representation as a (finite) sum $x = \sum_{g \in G} a_g g$ where $a_g = |x - g|$; also, if $g \in G$, then $|g| = 1$.

The object of this paper is to discuss some problems connected with congruent embeddings of $B$-metric spaces in a $B$-vector space with a finite basis. The results presented here resemble those of similar problems in connection with congruent embeddings of (ordinary) metric spaces in the Euclidean spaces. (See Ref. 1; part II, Chapter IV.) Further, they extend the same type of solutions of Melter for a $p$-ring to the case of any $B$-vector space with a finite basis. The proofs given here are much shorter than those of Melter even for the case of a $p$-ring.

One of the tools which we employ below is the concept of “inner product.” If $V$ is any normed $B$-vector space (with or without a basis) and $x, y \in V$, then $|x - y| \leq |x| + |y|$ so that there is a unique solution $(x, y)$ of the two equations $\eta |x - y| = 0$ and $\eta + |x - y| = |x| + |y|$. Also, if $V$ has a finite basis $g_1, g_2, \ldots, g_k$ and we write $x = \sum_{i=1}^{k} a_i g_i$ and $y = \sum_{i=1}^{k} b_i g_i$ where $x, y \in V$ and $a_i a_j = b_i b_j = 0$ for $i \neq j$, then we have $(x, y) = \sum_{i=1}^{k} a_i b_i$. The reader is referred to Refs. 4 and 5 for details of these and other concepts used in the presentation below.

2. AN ISOMETRY THEOREM

Let $V$ be a vector space with a finite basis $G$ over a Boolean algebra $B$; and let $k$ denote the number of elements in $G$. First we recall the following:

**Lemma 1:** If $y_1, y_2, \ldots, y_n \in V$, $n < k$ and $a > \sum_{i=1}^{n} |y_i|$, then there exists an element $z \in V$ such that $|z| = a$ and $|y_i - z| = a$ for $1 \leq i \leq n$.

**Corollary 1:** If $y_0, y_1, \ldots, y_n \in V$ and $n < k$, then there exists an element $y \in V$ such that $|y - y_i| = 1$ for $0 \leq i \leq n$.

**Lemma 2:** Any equilateral set contains at most $k + 1$ elements.