17. Cardinals in $V^{(\text{w})}$

The theorems of this section can be obtained from the corresponding results in the theory of forcing by translating them in the manner outlined in Corollary 14.23. However, since this translation requires the existence of elementary subsystems of $V$ and thus cannot be carried out in ZF, we shall try to give direct proofs in $V^{(\text{w})}$. Corresponding to the fact that every cardinal in $M[G]$, where $<M, P>$ is a setting for forcing and $G$ is $P$-generic over $M$, is a cardinal in $M$ we have the following.

**Theorem 17.1.** If $\alpha$ is not a cardinal, then $[\neg \text{Card} (\check{\alpha})] = 1$.

**Proof.** $\neg \text{Card} (\check{\alpha}) \iff (\exists f)(\exists \beta < \alpha)[f: \beta \rightarrow \alpha \land \mathcal{W}(f) = \alpha]$. Therefore $\neg \text{Card} (\check{\alpha}) \iff (\exists f)(\exists \beta < \alpha)\phi(f, \alpha)$ where $\phi(f, \alpha)$ is a bounded formula. Thus, by Corollary 13.18, if $\alpha$ is not a cardinal,

$$[\phi(f, \check{\alpha})] = 1$$

for some $f \in V$, hence

$$[\neg \text{Card} (\check{\alpha})] = 1.$$

**Remark.** As might be expected, for finite cardinals and for $\omega$ we can prove the converse of Theorem 17.1.

**Theorem 17.2.** For every $\alpha \leq \omega$, $[\text{Card} (\check{\alpha})] = 1$.

**Proof.** We have to show that

$$[\neg (\exists f)(\exists \beta < \check{\alpha})[f: \beta \rightarrow \check{\alpha} \land \mathcal{W}(f) = \check{\alpha}]] = 1$$

i.e.,

$$(\forall f \in V^{(\text{w})})(\forall \beta < \alpha)[[f: \check{\beta} \rightarrow \check{\alpha} \land \mathcal{W}(f) = \check{\alpha}] = 0].$$

Suppose that on the contrary,

$$b = [f: \check{\beta} \rightarrow \check{\alpha} \land \mathcal{W}(f) = \check{\alpha}] > 0$$

for some $f \in V^{(\text{w})}$, $\beta < \alpha$.

Then $b \leq [(\forall \eta < \check{\alpha})(\exists \xi < \check{\beta})[f(\check{\xi}) = \check{\eta}]],$

$$i \ (b \leq \prod_{\eta < \check{\alpha}} \sum_{\xi < \check{\beta}} [f(\check{\xi}) = \check{\eta}].$$

[Note: Let $\psi(\check{\xi}, \eta)$ be the formula that expresses $f(\check{\xi}) = \eta$. Then $[f(\check{\xi}) = \check{\eta}]$ means $[\psi(\check{\xi}, \check{\eta})].$]

$$\mathcal{W}(f) = \{y \mid (\exists x)(x, y) \in f\}.$$
Now let us assume that \( \alpha \leq \omega \). Since \( \beta < \alpha \),

\[
b \leq \prod_{\eta < \beta + 1} \sum_{\xi < \beta} [f(\xi) = \eta]
\]

\[
= \sum_{\varphi \in \beta(\beta + 1)} \prod_{\eta < \beta + 1} [f(\varphi(\eta)) = \eta]
\]

by the \((\beta + 1, \beta)\)-DL, (see Definition 4.1) which holds for every \( B \) since \( \beta \) is finite.

Therefore

\[
0 < b \prod_{\eta < \beta + 1} [f(\varphi(\eta)) = \eta] \quad \text{for some } \varphi: \beta + 1 \to \beta.
\]

There must exist \( n, m < \beta + 1 \) such that \( n \neq m \land \varphi(n) = \varphi(m) \). Then

\[
0 < b[f((\varphi(n))\gamma)] = \hat{n} \cdot [f((\varphi(m))\gamma)] = \hat{m}
\]

\[
\leq b[\hat{n} \cdot \hat{m}] \quad \text{since } b \leq [f((\varphi(n))\gamma)] = f((\varphi(m))\gamma)]
\]

\[
= 0 \quad \text{since } n \neq m.
\]

This is a contradiction.

**Remark.** It is easy to see that the same proof can be used to show: If \( B \) satisfies the \((\alpha, \alpha)\)-DL, where \( \alpha \) is a cardinal, then for each cardinal \( \gamma \leq \alpha \), \( \|\text{Card } (\gamma)\| = 1 \), i.e., cardinals \( \leq \alpha \) remain cardinals in \( V^{(B)} \). (It can also be shown that we only need the \((\alpha, 2)\)-DL since \((\alpha, 2)\)-DL \( \leftrightarrow \) \((\alpha, \alpha)\)-DL.)

In general, Theorem 17.2 does not hold for all cardinals. However, corresponding to Theorem 11.8 we have a converse of Theorem 17.1. For a more general result we introduce the following definition.

**Definition 17.3.** Let \( \gamma \) be a cardinal. A Boolean algebra \( B \) satisfies the \( \gamma \)-chain condition iff

\[
(\forall S \subseteq B)[(\forall \alpha, \gamma \in S)[\alpha \neq \gamma \to \alpha \cdot \gamma = 0] \to \overline{S} \leq \gamma].
\]

In particular, \( B \) satisfies the \( \omega \)-chain condition iff \( B \) satisfies the c.c.c.

**Theorem 17.4.** Let \( \gamma \) be an infinite cardinal and suppose that \( B \) satisfies the \( \gamma \)-chain condition. If \( \alpha > \gamma \) is a cardinal, then \( \|\text{Card } (\alpha)\| = 1 \).

**Proof.** As in the proof of Theorem 17.2, suppose that \( \|\text{Card } (\alpha)\| \neq 1 \) for some cardinal \( \alpha > \gamma \), then defining \( b \) as before, we have for some \( \beta < \alpha \), and \( f \in V^{(B)} \),

\[
i) \ b \leq \prod_{\eta < \alpha} \sum_{\xi < \beta} [f(\xi) = \eta] \quad \text{where } b > 0.
\]

Therefore, using the AC in \( V \),

\[
(\forall \eta < \alpha)(\exists \xi_n < \beta)[b \cdot [f(\xi_n) = \eta] \neq 0].
\]

For \( \xi < \beta \) define

\[
A_\xi = \{\eta < \alpha \mid \xi_n = \xi\}.
\]