CHAPTER IX

The Real and Complex Numbers

IX, §1. ORDERING OF RINGS

Let $R$ be an entire ring. By an ordering of $R$ one means a subset $P$ of $R$ satisfying the following conditions:

ORD 1. For every $x \in R$ we have $x \in P$, or $x = 0$, or $-x \in P$, and these three possibilities are mutually exclusive.

ORD 2. If $x, y \in P$ then $x + y \in P$ and $xy \in P$.

We also say that $R$ is ordered by $P$, and call $P$ the set of positive elements.

Let us assume that $R$ is ordered by $P$. Since $1 \neq 0$, and $1 = 1^2 = (-1)^2$ we see that $1$ is an element of $P$, i.e. $1$ is positive. By ORD 2 and induction, it follows that $1 + \cdots + 1$ (sum taken $n$ times) is positive. An element $x \in R$ such that $x \neq 0$ and $x \notin P$ is called negative. If $x, y$ are negative elements of $R$, then $xy$ is positive (because $-x \in P$, $-y \in P$, and hence $(-x)(-y) = xy \in P$). If $x$ is positive and $y$ is negative, then $xy$ is negative, because $-y$ is positive, and hence $x(-y) = -xy$ is positive. For any $x \in R$, $x \neq 0$, we see that $x^2$ is positive.

Suppose that $R$ is a field. If $x$ is positive and $x \neq 0$ then $xx^{-1} = 1$, and hence by the preceding remarks, it follows that $x^{-1}$ is also positive.

Let $R$ be an arbitrary ordered entire ring again, and let $R'$ be a subring. Let $P$ be the set of positive elements in $R$, and let $P' = P \cap R$. Then it is clear that $P'$ defines an ordering on $R'$, which is called the induced ordering.

More generally, let $R'$ and $R$ be ordered rings, and let $P'$, $P$ be their sets of positive elements respectively. Let $f: R' \to R$ be an embedding.
(i.e. an injective homomorphism). We shall say that \( f \) is \textbf{order-preserving} if for every \( x \in R' \) such that \( x \in P' \) we have \( f(x) \in P \). This is equivalent to saying that \( f^{-1}(P) = P' \) [where \( f^{-1}(P) \) is the set of all \( x \in R' \) such that \( f(x) \in P \)].

Let \( x, y \in R \). We define \( x < y \) (or \( y > x \)) to mean that \( y - x \in P \). Thus to say that \( x > 0 \) is equivalent to saying that \( x \in P \); and to say that \( x < 0 \) is equivalent to saying that \( x \) is negative, or \( -x \) is positive. One verifies easily the usual relations for inequalities, namely for \( x, y, z \in R \):

\begin{align*}
\text{IN 1. } & x < y \text{ and } y < z \quad \text{implies } x < z. \\
\text{IN 2. } & x < y \text{ and } z > 0 \quad \text{implies } xz < yz. \\
\text{IN 3. } & x < y \quad \text{implies } x + z < y + z.
\end{align*}

If \( R \) is a field, then

\[ \text{IN 4. } x < y \text{ and } x, y > 0 \quad \text{implies } 1/y < 1/x. \]

As an example, we shall prove \textbf{IN 2}. We have \( y - x \in P \) and \( z \in P \), so that by \textbf{ORD 2}, \((y - x)z \in P \). But \((y - x)z = yz - xz\), so that by definition, \( xz < yz \). As another example, to prove \textbf{IN 4}, we multiply the inequality \( x < y \) by \( x^{-1} \) and \( y^{-1} \) to find the assertion of \textbf{IN 4}. The others are left as exercises.

If \( x, y \in R \) we define \( x \leq y \) to mean that \( x < y \) or \( x = y \). Then one verifies at once that \textbf{IN 1, 2, 3} hold if we replace throughout the \(<\) sign by \(\leq\). Furthermore, one also verifies at once that if \( x \leq y \) and \( y \leq x \) then \( x = y \).

In the next theorem, we see how an ordering on an entire ring can be extended to an ordering of its quotient field.

**Theorem 1.1.** \textit{Let} \( R \) \textit{be an entire ring, ordered by} \( P \). \textit{Let} \( K \) \textit{be its quotient field. Let} \( P_K \) \textit{be the set of elements of} \( K \) \textit{which can be written in the form} \( a/b \) \textit{with} \( a, b \in R \), \( b > 0 \) \textit{and} \( a > 0 \). \textit{Then} \( P_K \) \textit{defines an ordering on} \( K \).

**Proof.** Let \( x \in K \), \( x \neq 0 \). Multiplying a numerator and denominator of \( x \) by \(-1\) if necessary, we can write \( x \) in the form \( x = a/b \) with \( a, b \in R \) and \( b > 0 \). If \( a > 0 \) then \( x \in P_K \). If \(-a > 0 \) then \(-x = -a/b \in P_K \). We cannot have both \( x \) and \(-x \in P_K \), for otherwise, we could write

\[ x = a/b \quad \text{and} \quad -x = c/d \]

with \( a, b, c, d \in R \) and \( a, b, c, d > 0 \). Then

\[ -a/b = c/d, \]