116. **Strong normal continuity of adjoint.** Even though the adjoint is not strongly continuous, it has an important continuous part.

**Problem 116.** *The restriction of the adjoint to the set of normal operators is strongly continuous.*

117. **Strong bounded continuity of multiplication.** The crux of the proof that multiplication is strongly sequentially continuous (Solution 113) is boundedness. That is: if \( \{A_n\} \) and \( \{B_n\} \) are nets that converge strongly to \( A \) and \( B \), respectively, and if \( \{\|A_n\|\} \) is bounded, then \( \{A_n B_n\} \) converges strongly to \( AB \). Is this result symmetric with respect to the interchange of right and left?

**Problem 117.** *If \( \{A_n\} \) and \( \{B_n\} \) are nets that converge strongly to 0, and if \( \{\|B_n\|\} \) is bounded, does it follow that \( \{A_n B_n\} \) converges strongly to 0?*

118. **Strong operator versus weak vector convergence.**

**Problem 118.** *If \( \{f_n\} \) is a sequence of vectors and \( \{A_n\} \) is a sequence of operators such that \( f_n \to f \) weakly and \( A_n \to A \) strongly, does it follow that \( A_n f_n \to Af \) weakly?*

119. **Strong semicontinuity of spectrum.** The spectrum of an operator varies upper semicontinuously (Problem 103). If, that is, an operator is replaced by one that is near it in the norm topology, then the spectrum can increase only a little. What if the strong topology is used in place of the norm topology?
PROBLEMS

Problem 119. *Is the spectrum strongly upper semicontinuous? What can be said about the spectral radius?*

120

120. Increasing sequences of Hermitian operators. A bounded increasing sequence of Hermitian operators is weakly convergent (to a necessarily Hermitian operator). To see this, suppose that \( \{A_n\} \) is an increasing sequence of Hermitian operators (i.e., \((A_n f, f) \leq (A_{n+1} f, f)\) for all \(n\) and all \(f\)), bounded by \(x\) (i.e., \((A_n f, f) \leq x\|f\|^2\) for all \(n\) and all \(f\)). If \(\psi_n(f) = (A_n f, f)\), then each \(\psi_n\) is a quadratic form. The assumptions imply that the sequence \(\{\psi_n\}\) is convergent and hence (Solution 1) that the limit \(\psi\) is a quadratic form. It follows that \(\psi(f) = (Af, f)\) for some (necessarily Hermitian) operator \(A\); polarization justifies the conclusion that \(A_n \to A\) (weakly).

Does the same conclusion follow with respect to the strong and the uniform topologies?

Problem 120. *Is a bounded increasing sequence of Hermitian operators necessarily strongly convergent? uniformly convergent?*

121

121. Square roots. The assertion that a positive operator has a unique positive square root is an easy consequence of the spectral theorem. In some approaches to spectral theory, however, the existence of square roots is proved first, and the spectral theorem is based on that result. The following assertion shows how to get square roots without the spectral theorem.

Problem 121. *If \(A\) is an operator such that \(0 \leq A \leq 1\), and if a sequence \(\{B_n\}\) is defined recursively by the equations

\[
B_0 = 0 \quad \text{and} \quad B_{n+1} = \frac{1}{2}((1 - A) + B_n^2), \quad n = 0, 1, 2, \ldots,
\]

then the sequence \(\{B_n\}\) is strongly convergent. If \(\lim_n B_n = B\), then \((1 - B)^2 = A\).*

122

122. Infimum of two projections. If \(E\) and \(F\) are projections with ranges \(M\) and \(N\), then it is sometimes easy and sometimes hard to find, in terms of \(E\) and \(F\), the projections onto various geometric constructs formed with \(M\) and \(N\). Things are likely to be easy if \(E\) and \(F\) commute. Thus, for instance, if \(M \subseteq N\), then it is easy to find the projection with range \(N \cap M^2\). and if \(M \perp N\), then it is easy to find the projection with range \(M \vee N\). In the absence of such special assumptions, the problems become more interesting.

Problem 122. *If \(E\) and \(F\) are projections with ranges \(M\) and \(N\), find the projection \(E \wedge F\) with range \(M \cap N\).*