8
Propagating of Chaos

8.1 Introduction

This chapter is concerned with propagation-of-chaos properties of particle models. These properties measure the adequacy of the laws of the particles with the desired limiting distribution. They also allow us to quantify the independence between particles. Loosely speaking, the initial configuration of an $N$-particle model consists of $N$ independent particles in a "complete chaos." Then they evolve and interact with one another. The nature of the interactions depends on the McKean interpretation of the limiting process (see Section 2.5.3). For any fixed time horizon $n$, when the size of the system $N$, tends to infinity, any finite block of $q(\leq N)$ particles asymptotically behaves as a collection of independent particles. In other words, the law of any $q$ particle paths of length $n$ converges as $N \to \infty$ towards the $q$ tensor product of the $n$-path McKean measure.

The interpretations of propagation of chaos differ from the different application particle model areas we consider.

From the physical point of view, particle algorithms are often related to some microscopic particle interpretation of some physical evolution equation. In this context, the limiting distribution flow model is regarded as an infinite particle model. Here propagation-of-chaos estimates give precise information on the degree of interaction between the particles. They justify in some sense the well-founded microscopic particle interpretations.

From a statistical point of view, particle methods are rather regarded as particle simulation techniques of complex path distributions. In this
context, propagation-of-chaos properties offer precise information on the numerical quality of these simulation techniques. First of all, they make it possible to quantify independence between the simulated variables. Moreover, they guarantee the adequacy of their laws with the desired target distribution. For instance, in engineering applications such as in nonlinear filtering or global optimization problem, propagation of chaos ensures the adaptation of the stochastic grid with the signal conditional distributions or the Boltzmann-Gibbs concentration laws.

From the biological perspective, the propagation-of-chaos of genetic models gives precise information on their genealogical structure. More precisely, they quantify the degree of interaction between the ancestral lines of evolution of a group of individuals. They provide information not only on current populations but also on the complete genealogies of ancestral lines that have disappeared.

We design three strategies with different precision levels. In the first one, we examine the propagation of chaos of the particle model associated to the McKean interpretation model

\[
K_{n+1,\eta}(x, \cdot) = \varepsilon_n G_n(x) M_{n+1}(x, \cdot) + (1 - \varepsilon_n G_n(x)) \Phi_{n+1}(\eta)
\]  

(8.1)

where \( \varepsilon_n \) are nonnegative constants such that \( \varepsilon_n G_n \leq 1 \). Note that the pair of examples provided on page 219 fit into this model, and the case \( \varepsilon_n = 0 \) corresponds to the traditional mutation/selection genetic algorithm. We present a general and basic strategy that probably works for other McKean interpretations but does not give any information on the rate of propagation of chaos. Another drawback of this technique is that it is restricted to locally compact and separable metric state spaces.

One important question arising in practice is to estimate the rate of propagation of chaos with respect to the pair parameters \((q, n)\). This led for instance to propagation-of-chaos properties with respect to increasing particle block sizes and/or time horizons. We first derive strong propagation-of-chaos estimates with respect to the relative entropy criterion. This strategy is based on an inequality of Csiszar on exchangeable measures. It allows us to restrict the analysis to profile measures. The only drawback of this elegant entropy technique is that it requires some regularity on the mutation transitions. As a result, it doesn’t apply to path-space and genealogical tree models.

The third strategy is not based on any kind of regularity property of the Feynman-Kac model but it is restricted, as presented, to the simple mutation/selection genetic model. We use as a tool a natural tensor product Feynman-Kac semigroup approach with respect to time horizons and particle block sizes. We derive several propagation-of-chaos estimates for Boltzmann-Gibbs measures from a precise moment analysis of empirical measures and from an original transport equation relating q-tensor product and symmetric statistic type empirical measures. This analysis applies to the study of the asymptotic behavior of genetic historical processes and