Central Limit Theorems

9.1 Introduction

The central limit theorem (abbreviated CLT) is one of the most startling results in probability theory. Loosely speaking, it expresses the fact that the sums of local and small independent disturbances (with finite variances) behave asymptotically, at least as Gaussian variables. The first CLT was stated and proved for symmetric and Bernoulli independent disturbances by A. De Moivre in the 18th century (Miscellanea analytica supplementum, 1730). This result was extended by P.S. Laplace in 1812 to general Bernoulli trials in his celebrated treatise *Théorie analytique des probabilités*.

These two pioneering studies have been further developed by several mathematicians, such as, in alphabetical order, Donsker, Dynkin, Feller, Jacod, Lindeberg, Lyapunov, Mandelbaum, and Shiryaev. These developments were followed in various directions going from multidimensional models, symmetric statistics sequences, and empirical processes to fairly general classes of nonidentically distributed and dependent random variables such as properly scaled random triangular arrays or martingale sequences. There also exist several approximation tools, such as the so-called $\delta$-method or Slutsky's technique, to deduce various weak limit results from a given CLT. It is of course out of the scope of this book to review in detail all of these developments, and rather we refer the interested reader to any classical textbook on probability and limit theorems for stochastic processes.

Most of these CLTs are derived using extended versions of the celebrated Levy's convergence theorem, which basically says that the weak conver-
gence of distributions corresponds exactly to the pointwise convergence of their characteristic functions. In this connection, we also mention that the inequality of Berry and Esseen provides an estimation of the maximum difference between distribution functions by means of an average difference between their characteristic functions. This inequality allows us to quantify the speed of convergence of the CLT.

The aim of this chapter is to extend these results to interacting particle approximation models. These fluctuation results provide precise asymptotic information on the various $L_p$ mean error bounds and the propagation of chaos derived respectively in Chapter 7 and Chapter 8. The reader will find a deep study of topics such as the multidimensional CLTs for normalized and unnormalized particle approximation measures, extended versions of the theorems of Berry and Esseen and of Donsker to interacting particle models, and a fluctuation result for particle McKean measures on path space. While our approach to CLTs is well suited to analyze any sufficiently regular McKean interpretation model, to simplify the presentation, we restrict our study to McKean models of the form

$$K_{n+1,\eta}(x, \cdot) = \varepsilon_n G_n(x) M_{n+1}(x, \cdot) + (1 - \varepsilon_n G_n(x)) \Phi_{n+1}(\eta)$$

(9.1)

where $\varepsilon_n$ are nonnegative constants such that $\varepsilon_n G_n \leq 1$. As in Chapter 8, we note that the pair of examples provided on page 219 can be cast in the form above, and the case $\varepsilon_n = 0$ corresponds to the traditional mutation/selection genetic algorithm. As usual, we shall suppose that the potential functions $G_n$ satisfy the traditional condition $(G)$ introduced in page 115 for some sequence of parameters $\varepsilon_n(G) > 0$.

The chapter is organized as follows. Section 9.3 and Section 9.4 focus on multidimensional CLTs for particle density profiles. We examine the particle approximation models associated with unnormalized and normalized Feynman-Kac measures. We design an elegant strategy based on martingales and semigroups techniques which allows us to reduce the fluctuation analysis to local sampling errors. In Section 9.5 we study the rate of convergence of these CLTs and provide a Berry-Esseen type theorem for abstract martingale sequences and interacting processes. We provide simple regularity conditions on the increasing processes under which we can derive precise estimations of the characteristic functions. In Section 9.6, we discuss the fluctuations of particle random fields and prove an extended version of Donsker’s theorem for interacting particle models. We develop an empirical process technique based on sub-Gaussian maximal inequalities to prove the asymptotic tightness of random field errors. All of these fluctuation results for interacting processes are restricted to uniformly bounded classes of test functions. Some extensions to unbounded functions can be found in [89]. We emphasize that the strategy for CLTs presented in Section 9.4 to Section 9.6 is not related to any kind of regularity conditions on the mutation transition $M_n$. Thus, it applies to path space and genealogical particle models. In Section 9.7, we analyze the fluctuations of particle