Differentiation of real valued functions

We now begin the differential calculus for real valued functions of several variables. The first step is to define the notions of directional derivative and partial derivative. Then the concept of differentiable function is introduced, by linear approximation to the increments of a function. Taylor's formula with remainder is obtained for functions of class $C^q$; such functions have continuous partial derivatives of orders $1, 2, \ldots, q$. It is then applied to problems of relative extrema and to the characterization of convex functions of class $C^2$.

The chain rule for partial derivatives is postponed to Chapter 4, since it is a natural corollary of the composite function theorem for vector-valued functions to be proved there.

3.1 Directional and partial derivatives

If $f$ is a function of one variable, then its derivative at a point $x_0$ is defined by

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists. The corresponding expression for functions of several variables does not make sense, since $h$ is then a vector and division by $h$ is undefined. Therefore we must find an acceptable substitute for it. Let us first consider the derivative of $f$ in various directions.

Let us call any unit vector $v$ (that is, vector with $|v| = 1$) a direction in $E^n$. The directions are just the points of the $(n - 1)$-dimensional sphere which bounds the unit $n$-ball. If $n = 1$, the only directions are $e_1$ and $-e_1$, which we have identified with the scalars 1 and -1. If $n = 2$, every direction can be written $(\cos \theta, \sin \theta)$ where $0 \leq \theta < 2\pi$. The angle $\theta$ determines the
direction. For any \( n \geq 2 \) the components of a direction \( v \) satisfy \( v^i = \cos \theta_i, \)
\( i = 1, \ldots, n, \) where \( \theta_i \) is the angle between \( v \) and \( e_i. \)

Given \( x_0 \) and a direction \( v, \) the line through \( x_0 + v \) and \( x_0 \) is called the line through \( x_0 \) with direction \( v. \) According to the definition in Section 1.3, this line is

\[
\{ x : x = x_0 + tv, \quad t \text{ any scalar} \}.
\]

Let \( f \) be a function with domain \( D \subset \mathbb{R}^n, \) and let \( x_0 \) be an interior point of \( D. \)

**Definition.** The **derivative of \( f \) at \( x_0 \) in the direction \( v \)** is

\[
(3.2) \quad \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}
\]

if the limit exists.

Since \( x_0 \) is an interior point, the \( \delta \)-neighborhood of \( x_0 \) is contained in \( D \) for some \( \delta > 0 \) (Figure 3.1). Since

\[
|(x_0 + tv) - x_0| = |tv| = |t|,
\]

\( x_0 + tv \in D \) provided \( |t| < \delta. \) The domain of the function \( \phi \) defined by

\[
\phi(t) = f(x_0 + tv)
\]

contains the \( \delta \)-neighborhood of \( 0. \) The derivative of \( f \) in the direction \( v \) is \( \phi'(0), \) if \( \phi \) has a derivative at \( 0. \)

The line through \( x_0 \) with direction \( -v \) is the same line as the one through \( x_0 \) with direction \( v. \) However, the derivative in the direction \( -v \) is the negative of the derivative in direction \( v \) (Problem 6). The direction \( v \) defines an **orientation** of this line, and \( -v \) the opposite orientation. When the orientation changes, the directional derivative changes sign. In effect, by assigning the orientation \( v \) we agree that the point \( x_0 + sv \) precedes \( x_0 + tv \) on the line if \( s < t. \)

![Figure 3.1](image-url)