1.1 Introduction

Topology is an abstraction of geometry; it deals with sets having a structure which permits the definition of continuity for functions and a concept of "closeness" of points and sets. This structure, called the "topology" on the set, was originally determined from the properties of open sets in Euclidean spaces, particularly the Euclidean plane.

It is assumed in this text that the reader has some familiarity with basic topology, including such concepts as open and closed sets, compactness, connectedness, metrizability, continuity, and homeomorphism. All of these are normally studied in what is called "point-set topology"; an outline of the prerequisite information is contained in Appendix 2.

Point-set topology was strongly influenced by the general theory of sets developed by Georg Cantor around 1880, and it received its primary impetus from the introduction of general metric spaces by Maurice Frechet in 1906 and the appearance of the book *Grundzüge der Mengenlehre* by Felix Hausdorff in 1912.

Although the historical origins of algebraic topology were somewhat different, algebraic topology and point-set topology share a common goal: to determine the nature of topological spaces by means of properties which are invariant under homeomorphisms. Algebraic topology describes the structure of a topological space by associating with it an algebraic system, usually a group or a sequence of groups. For a space $X$, the associated group $G(X)$ reflects the geometric structure of $X$, particularly the arrangement of the "holes" in the space. There is a natural interplay between continuous maps $f : X \to Y$ from one space to another and algebraic homomorphisms $f^* : G(X) \to G(Y)$ on their associated groups.
Consider, for example, the unit circle $S^1$ in the Euclidean plane. The circle has one hole, and this is reflected in the fact that its associated group is generated by one element. The space composed of two tangent circles (a figure eight) has two holes, and its associated group requires two generating elements.

The group associated with any space is a topological invariant of that space; in other words, homeomorphic spaces have isomorphic groups. The groups thus give a method of comparing spaces. In our example, the circle and figure eight are not homeomorphic since their associated groups are not isomorphic.

Ideally, one would like to say that any topological spaces sharing a specified list of topological properties must be homeomorphic. Theorems of this type are called classification theorems because they divide topological spaces into classes of topologically equivalent members. This is the sort of theorem to which topology aspires, thus far with limited success. The reader should be warned that an isomorphism between groups does not, in general, guarantee that the associated spaces are homeomorphic.

There are several methods by which groups can be associated with topological spaces, and we shall examine two of them, homology and homotopy, in this course. The purpose is the same in each case: to let the algebraic structure of the group reflect the topological and geometric structures of the underlying space. Once the groups have been defined and their basic properties established, many beautiful geometric theorems can be proved by algebraic arguments. The power of algebraic topology is derived from its use of algebraic machinery to solve problems in topology and geometry.

The systematic study of algebraic topology was initiated by the French mathematician Henri Poincaré (1854–1912) in a series of papers during the years 1895–1901. Algebraic topology, or analysis situs, did not develop as a branch of point-set topology. Poincaré's original paper predated Fréchet's introduction of general metric spaces by eleven years and Hausdorff's classic treatise on point-set topology, *Grundzüge der Mengenlehre*, by seventeen years. Moreover, the motivations behind the two subjects were different. Point-set topology developed as a general, abstract theory to deal with continuous functions in a wide variety of settings. Algebraic topology was motivated by specific geometric problems involving paths, surfaces, and geometry in Euclidean spaces. Unlike point-set topology, algebraic topology was not an outgrowth of Cantor's general theory of sets. Indeed, in an address to the International Mathematical Congress of 1908, Poincaré referred to point-set theory as a "disease" from which future generations would recover.

Poincaré shared with David Hilbert (1862–1943) the distinction of being the leading mathematician of his time. As we shall see, Poincaré's geometric

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1 The papers were *Analysis Situs*, *Complément à l'Analysis Situs*, *Deuxième Complément*, and *Cinquième Complément*. The other papers in this sequence, the third and fourth complements, deal with algebraic geometry.