CHAPTER 17
Division Algebras over Local Fields

This chapter gives a fairly complete description of the finite dimensional division algebras over fields that are locally compact in the topology of a discrete valuation, that is, local fields. The most important property of these algebras is that they contain maximal subfields that are unramified extensions of their centers. It follows that all such algebras are cyclic. Moreover, the classification of the unramified extensions of local fields gives a characterization of the Brauer groups of such fields; they are all isomorphic to \( \mathbb{Q}/\mathbb{Z} \).

The theory of field valuations can be extended in a straightforward way to division algebras. The first half of this chapter gives a self-contained development of this subject. No prior knowledge of valuation theory is assumed.

17.1. Valuations of Division Algebras

The basic definitions in the theory of valuations are given in this section. Our main result relates the valuations of a division algebra \( D \in \mathcal{D}(F) \) to the reduced norm \( v_{DF} \).

**Definition.** A *valuation* of a division algebra \( D \) is a mapping \( v : D \to \mathbb{R} \) such that

\[
\begin{align*}
v(x) &\geq 0 \text{ for all } x \in D \text{ and } v(x) = 0 \text{ if and only if } x = 0, \\
v(xy) &= v(x)v(y) \text{ for all } x, y \in D, \\
\text{there is a positive real number } a \text{ such that } v(x + y) &\leq a \max\{v(x), v(y)\} \text{ for all } x, y \in D.
\end{align*}
\]

(1) (2) (3)

R. S. Pierce, *Associative Algebras*  
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If \( v \) is a valuation of \( D \), then \( v|D^\circ \) is a homomorphism of \( D^\circ \) to the multiplicative group \( \mathbb{R}^+ \) of positive real numbers. Conversely, a homomorphism \( v: D^\circ \to \mathbb{R}^+ \) can be extended to a valuation by \( v(0) = 0 \) if (3) is satisfied. For example, the homomorphism of \( D^\circ \) to \( \{1\} \) gives a valuation \( v \) such that \( v(x) = 1 \) for all \( x \neq 0 \) and \( v(0) = 0 \). This \( v \) is called the trivial valuation of \( D \). Other valuations of \( D \) are called non-trivial.

The uniqueness of roots in \( \mathbb{R}^+ \) implies that if \( v(x^n) = v(y^n) \), then \( v(x) = v(y) \). In particular, if \( x \) is a root of unity in \( D \), then \( v(x) = v(1_D) = 1 \). Two special cases of this observation are used frequently: \( v(-x) = v(x) ; v(-1_D) = 1 \).

By (3), the set \( \{v(1+x) : x \in D, v(x) \leq 1\} \) is bounded. Define

\[
a(v) = \sup\{v(1+x) : x \in D, v(x) \leq 1\}.
\]

Lemma a. If \( v \) is a valuation of the division algebra \( D \) and \( x_1, x_2, \ldots, x_n \in D \), then

\[
v(x_1 + x_2 + \cdots + x_n) \leq a(v)^m \max\{v(x_i) : 1 \leq i \leq n\},
\]

where \( m \) is the least integer greater than \( \log_2 n \).

An easy argument gives this inequality for \( n = 2 \). Induction extends it to powers of 2. The final form of the lemma is obtained by adjoining \( 2^m - n \) zeros to the sum \( x_1 + \cdots + x_n \).

If \( v \) is a valuation of the division algebra \( D \) and \( e \in \mathbb{R}^+ \), then the mapping \( v^e : D \to \mathbb{R} \) defined by \( v^e(x) = v(x)^e \) is also a valuation of \( D \). Moreover, since \( v^e(x) \leq 1 \) if and only if \( v(x) \leq 1 \), it follows from (4) that

\[
a(v^e) = a(v)^e.
\]

Two valuations \( v \) and \( w \) of the division algebra \( D \) are equivalent if \( w = v^e \) for some \( e \in \mathbb{R}^+ \). Since \( (v^e)^f = v^{ef} \), the concept of equivalence for valuations is an equivalence relation on the set of valuations of \( D \).

Lemma b. Let \( v \) be a valuation of the division algebra \( D \).

(i) \( a(v) \geq 1 \).
(ii) If \( a(v) = 1 \), then \( a(w) = 1 \) for all valuations \( w \) that are equivalent to \( v \).
(iii) If \( a(v) > 1 \) and \( 1 < a \in \mathbb{R} \), then there is a unique valuation \( w \) such that \( w \) is equivalent to \( v \) and \( a(w) = a \).

The property (i) is clear from (4) because \( v(0) = 0 \) and \( v(1) = 1 \); the statements (ii) and (iii) follow easily from (5).

Proposition. If \( v \) is a valuation of the division algebra \( D \), then \( a(v) \leq 2 \) if and only if \( v \) satisfies the triangle inequality: \( v(x + y) \leq v(x) + v(y) \) for all \( x, y \in D \). Every valuation of \( D \) is equivalent to a valuation that satisfies the triangle inequality.