STRUCTURALLY INVARIANT LINEAR MODELS OF STRUCTURALLY VARYING LINEAR SYSTEMS

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I. INTRODUCTION

In the process of model building, assumptions leading to linearity are routinely made since the resulting models can be thoroughly understood using analytical techniques. Another assumption that is also very commonly used along with linearity is the assumption of invariance, or uniformity, of structure with respect to some set of supporting variables which generally represent time (time-invariance) or space (space-invariance). The assumption of linearity and invariance permits harmonic analysis to be applied. Understanding model behavior and sensitivity to parameter settings is greatly simplified by using frequency domain representations, i.e., Fourier transforms, of appropriate functions.

It is important to remember that assumptions of linearity and invariance are actually assumptions and that their justification and validity can be crucial to the success of a model. The difficulties inherent in making these assumptions need to be understood even though they allow such powerful mathematical theories to be applied. The linearization of non-linear systems has been studied to the extent that its domain of applicability is fairly clearly delineated. On the other hand, the assumption that a system which is not actually invariant can be modelled by a time or space invariant linear system has received relatively little attention.

Results exist for slowly-varying systems and for periodically-varying systems [1], [2], [3], but these kinds of variability do not model what is tacit in every assumption of invariance; namely, the intuition that small-scale temporal or
spatial variation, however rapid, can be ignored since it probably does not affect large-scale behavior. As long as fine behavioral detail is not measurable or is not desired, any fine variation in structural detail can, it is felt, be ignored without resulting in a misleading model. This assumption is invoked, for example, when a time-invariant model is constructed on the basis of observing responses to impulsive input applied at various times and noting that for the resolution level of the measuring instruments these responses appear to have identical shapes. Variation unobservable at this resolution level is tacitly assumed to be irrelevant. This is not the same as "slow" variation which can be approximated by constancy over short spatial or temporal distance.

In this paper we examine conditions under which the intuition that the effects of fine structural variation are confined to fine behavioral variation can be justified. We show that it is indeed true, for a particular characterization of small-scale variation, that this intuition can be rigorously defended. On the other hand, for an equally satisfying characterization of such variation, we illustrate that structural variation, however high a resolution level is required to observe it, can influence aspects of behavior which are observable at lower resolution levels. It can be concluded that non-trivial assumptions are required to justify structurally invariant linear models.

The development of these results depends on the concept of system homomorphism as a formalization of what it means for a model to be a non-misleading representation of a system. We shall think of a "real" system as actually being validly modelled by one of a class of linear systems having specifiable types of structural variation. We then seek structurally constant linear systems which are homomorphic images of the variable system. Using the terminology of Zeigler [4], we assume a "base model" in the form of a structurally varying system and examine conditions which justify forming constant "lumped models" such that structure and/or behavior preserving morphisms exist between the two classes of models.

II. NOTATION

Structural invariance, and its lack, will be defined group-theoretically in a manner which can be specialized to both time and space invariance. Let \( G \) be a group under an operation denoted by \( '+' \) and let \( K \) be a field (the results to follow also hold for \( K \) a suitable integral domain or finitely generated module [5]). The set of all functions from \( G \) into \( K \), denoted by \( F(G,K) \), is a linear space under the usual definitions of function addition and multiplication by scalars. For temporal interpretations, \( G \) is