Simple Problems in One Dimension

Now that the postulates have been stated and explained, it is all over but for the applications. We begin with the simplest class of problems—concerning a single particle in one dimension. Although these one-dimensional problems are somewhat artificial, they contain most of the features of three-dimensional quantum mechanics but little of its complexity. One problem we will not discuss in this chapter is that of the harmonic oscillator. This problem is so important that a separate chapter has been devoted to its study.

5.1. The Free Particle

The simplest problem in this family is of course that of the free particle. The Schrödinger equation is

\[ i\hbar \psi = H \psi = \frac{p^2}{2m} \psi \]

(5.1.1)

The normal modes or stationary states are solutions of the form

\[ |\psi\rangle = |E\rangle e^{-iEt/\hbar} \]

(5.1.2)

Feeding this into Eq. (5.1.1), we get the time-independent Schrödinger equation for \(|E\rangle\):

\[ H|E\rangle = \frac{p^2}{2m} |E\rangle = E|E\rangle \]

(5.1.3)

This problem can be solved without going to any basis. First note that any eigenstate
of $P$ is also an eigenstate of $P^2$. So we feed the trial solution $|p\rangle$ into Eq. (5.1.3) and find

\[
\frac{p^2}{2m} |p\rangle = E |p\rangle
\]

or

\[
\left(\frac{p^2}{2m} - E\right) |p\rangle = 0
\]  
(5.1.4)

Since $|p\rangle$ is not a null vector, we find that the allowed values of $p$ are

\[
p = \pm (2mE)^{1/2}
\]  
(5.1.5)

In other words, there are two orthogonal eigenstates for each eigenvalue $E$:

\[
|E, + \rangle = |p = (2mE)^{1/2}\rangle
\]  
(5.1.6)

\[
|E, - \rangle = |p = -(2mE)^{1/2}\rangle
\]  
(5.1.7)

Thus, we find that to the eigenvalue $E$ there corresponds a degenerate two-dimensional eigenspace, spanned by the above vectors. Physically this means that a particle of energy $E$ can be moving to the right or to the left with momentum $|p| = (2mE)^{1/2}$. Now, you might say, “This is exactly what happens in classical mechanics. So what’s new?” What is new is the fact that the state

\[
|E\rangle = \beta |p = (2mE)^{1/2}\rangle + \gamma |p = -(2mE)^{1/2}\rangle
\]  
(5.1.8)

is also an eigenstate of energy $E$ and represents a single particle of energy $E$ that can be caught moving either to the right or to the left with momentum $(2mE)^{1/2}$!

To construct the complete orthonormal eigenbasis of $H$, we must pick from each degenerate eigenspace any two orthonormal vectors. The obvious choice is given by the kets $|E, + \rangle$ and $|E, - \rangle$ themselves. In terms of the ideas discussed in the past, we are using the eigenvalue of a compatible variable $P$ as an extra label within the space degenerate with respect to energy. Since $P$ is a nondegenerate operator, the label $p$ by itself is adequate. In other words, there is no need to call the state $|p, E = P^2/2m\rangle$, since the value of $E = E(p)$ follows, given $p$. We shall therefore drop this redundant label.

The propagator is then

\[
U(t) = \int_{-\infty}^{\infty} |p\rangle\langle p| e^{-iE(p)t - \hbar} dp
\]

\[
= \int_{-\infty}^{\infty} |p\rangle\langle p| e^{-ip^2t/2m - \hbar} dp
\]  
(5.1.9)