CHAPTER 1

The Topological Point of View

This book is about the topological approach to certain topics in analysis, but what does that really mean? Starting with the "epsilon - delta" parts of elementary calculus, analysis makes extensive use of topological ideas and techniques. Thus the issue is not whether analysis requires topology, but rather how central a role the topological material plays. Rather than attempt the hopeless task of defining precisely what I mean by the topological point of view in analysis, I'll illustrate it by outlining two proofs of a well-known theorem about the existence of solutions to ordinary differential equations. In the first proof, the key step is the construction of a sequence of approximate solutions whose limit is the required solution. In the second proof, a general topological theorem about the behavior of self-maps of linear spaces implies the existence of the solution. The two proofs have several features in common, including their dependence on a substantial topological result, but I trust that even my (intentionally) very sketchy treatment will make it clear how basic the differences are in the ways that the two arguments reach the same conclusion. Here's the theorem.

Theorem 1.1. Cauchy-Peano Existence Theorem. Given a function \( f : \mathbb{R}^2 \to \mathbb{R} \) which is continuous in a neighborhood of a point \((x_0, y_0) \in \mathbb{R}^2\), there exists \( \alpha > 0 \) and a solution to the initial-value problem

\[
y' = f(x, y) \quad y(x_0) = y_0
\]

on the interval \([x_0 - \alpha, x_0 + \alpha]\). That is, there exists a continuous function \( \phi : [x_0 - \alpha, x_0 + \alpha] \to \mathbb{R} \) such that \( \phi(x_0) = y_0 \) and \( \phi'(x) = f(x, \phi(x)) \) for all \( x \) in the interval.

The two proofs produce the number \( \alpha \) in the same way. Since \( f \) is continuous in a neighborhood of \((x_0, y_0) \in \mathbb{R}^2\), there exist \( a, b > 0 \) such that if \((x, y) \in \mathbb{R}^2 \) with \(|x - x_0| \leq a \) and \(|y - y_0| \leq b\), then \( f \) is continuous at \((x, y)\). Let \( R \) be the rectangle in the plane consisting of such points, that is,

\[
R = \{(x, y) \in \mathbb{R}^2 : |x - x_0| \leq a \quad \text{and} \quad |y - y_0| \leq b\}
\]

and choose \( M > 1 \) such that \( M > \frac{b}{a} \) and also \( M \geq |f(x, y)| \) for all \((x, y) \in R\). Then set \( \alpha = \frac{b}{M} \). Notice how the definition of \( \alpha \) depended on some
familiar topology. A neighborhood means an open set and therefore the euclidean topology of the plane gives us an open disc about \((x_0, y_0)\) on which \(f\) is continuous. We choose \(a\) and \(b\) small enough to fit the rectangle \(R\) inside the open disc. We know that the set of values \(|f(x, y)|\) for \((x, y) \in R\) is bounded, and therefore \(M\) exists, because \(R\) is closed and bounded, that is, compact, so by a standard result its image under the continuous function \(|f|\) is a compact subset of the line and therefore bounded.

Another feature the two proofs have in common is that they make use of the fact that the fundamental theorem of calculus gives, as an equivalent form of the initial-value problem, the integral equation

\[
y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt.
\]

That is, a function \(\phi : [x_0 - \alpha, x_0 + \alpha] \to \mathbb{R}\) is a solution to the initial-value problem if and only if it is a solution to the integral equation.

The remaining common feature is that substantial topological result I referred to earlier: the Ascoli-Arzela theorem. I’ll indicate in both proofs where and how this theorem is used, but in neither case is it necessary to state the result itself. However, I’ll present a detailed discussion and proof of this theorem in the next chapter because the Ascoli-Arzela theorem will play a crucial role throughout the entire book.

1.1. Outline of the Approximation Proof.

For each integer \(n \geq 1\), choose \(\delta_n > 0\) small enough so that \(|x - \bar{x}| < \delta_n\) and \(|y - \bar{y}| < \delta_n\) implies

\[
|f(x, y) - f(\bar{x}, \bar{y})| < \frac{1}{n}.
\]

Then choose points

\[
x_0 - \alpha = x_{-k_n} < x_{-k_n+1} < \cdots < x_{-1} < x_0 < x_1 < \cdots < x_{k'-1} \leq x_{k_n}
\]

such that

\[
|x_{j+1} - x_j| \leq \frac{\delta_n}{M}.
\]

Define a piecewise-linear, continuous function \(\phi_n : [x_0 - \alpha, x_0 + \alpha] \to \mathbb{R}\) in the following manner. See Figure 1. On the interval \([x_0, x_1]\), set \(\phi_n(x_0) = y_0\) and let the slope of the line segment equal \(f(x_0, y_0)\). On the interval \([x_1, x_2]\), the slope of the line segment is \(f(x_1, y_1)\), where \(y_1 = \phi_n(x_1)\). Continue in this manner, moving to the right until you