17. Vector Fields, Flows, and 1-Forms

The equations of mathematical physics are typically ordinary or partial differential equations for vector or tensor fields over Riemannian manifolds whose group of isometries is a Lie group. It is taken as axiomatic that the equations be independent of the observer, in a sense we shall make precise below; and the consequence of this axiom is that the equations are invariant with respect to the group action. The action of a Lie group on tensor fields over a manifold is thus of primary importance. The action of a Lie group on a manifold $M$ induces in a natural way automorphisms of the algebra of $C^\infty$ functions over $M$ and on the algebra of tensor fields over $M$. The one parameter subgroups of the group induce one parameter subgroups of automorphisms of the tensor fields. The infinitesimal generators of these groups of automorphisms are the Lie derivatives of the action.

For example the Navier–Stokes equations for a viscous incompressible fluid are

$$\Delta u^i - \frac{\partial p}{\partial x^i} = u^j \frac{\partial u^i}{\partial x^j} + \frac{\partial u^i}{\partial t}$$

$$\frac{\partial u^i}{\partial x^i} = 0.$$  

(We use the summation convention here.) This is a system of equations for a vector field $u = (u^i)$ and scalar $p$. One may ask how these equations transform under rigid motions of the underlying space $\mathbb{R}^3$. In fact, one finds that these equations are equivariant with respect to the group of rigid motions $SO(3)$ of $\mathbb{R}^3$ in the following sense. A rigid motion in $\mathbb{R}^3$ is given by $\{O, a\}X = OX + a$...
where $O \in O(3)$ and $a \in \mathbb{R}^3$. A representation of $O(3)$ on the four component quantity $w = (u(x), p(x))$ is given by

$$T_g \begin{pmatrix} u^1 \\ u^2 \\ u^3 \\ p \end{pmatrix}(x) = \begin{pmatrix} 0_{11} & 0_{12} & 0_{13} & 0 \\ 0_{21} & 0_{22} & 0_{23} & 0 \\ 0_{31} & 0_{32} & 0_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \\ u^3 \\ p \end{pmatrix}(g^{-1}x)$$

where $g = \{O, a\}$. Now let $G = (G^1, G^2, G^3, G^4)$ be the four components of the Navier–Stokes equations; that is, $G^i = \Delta u^i - \frac{\partial p}{\partial x^i} - u^j \frac{\partial u^i}{\partial x^j}$ for $i = 1, 2, 3$, and $G^4 = \sum_{i=1}^{3} \frac{\partial u^i}{\partial x^4}$. The Navier–Stokes equations are equivariant with respect to the group of rigid motions in the sense that

$$T_g G(w) = G(T_g w).$$

This mathematical condition is a statement of the fact that the Navier–Stokes equations are the same in any Euclidean frame of reference.

As a second example, the electron has an internal structure due to its spin, and so is represented by a vector valued wave function known as a spinor. In the non-relativistic theory the spinor has two components. We saw in Chapter 1 that a matrix $A \in SU(2)$ generates a rotation $R(A)$ in $SO(3)$. Under the action of a rotation $R$, $x' = Rx$, and the spinor $\psi$ transforms according to the rule

$$\psi'(x') = e^{-i(\theta/2)\hat{n} \cdot \hat{\sigma}} \psi(x)$$

where $R$ is the rotation through an angle $\theta$ about the axis $\hat{n}$. Note that $(T_A \psi)(x) = A\psi(R^{-1}(A)x)$ is a representation of $SU(2)$. In Dirac's relativistic theory the electron the spinor has 4 components and transforms according to a representation of the Lorentz group. The Dirac equations of the electron are equivariant with respect to that representation. Alternatively, we may say the Dirac equations are invariant under Lorentz transformations.

In §17 and §18 we present a quick introduction to the basic ideas of tensor calculus on manifolds, turning, in §19, to the transformations of tensor fields induced by group actions on the manifold. We begin with an introduction to vector fields and flows in $\mathbb{R}^n$ and then discuss the situation on a general differentiable manifold.

Let us begin with a simple example. Consider the transformations of the plane given by

$$\varphi_t(x, y) = (x \cos t + y \sin t, -x \sin t + y \cos t).$$

The family $\{\varphi_t\}$ constitutes a one parameter group of transformations of the plane, since, as we easily see, $\varphi_{t+s} = \varphi_t \circ \varphi_s$. Such a one-parameter transformation group is called a flow on $\mathbb{R}^2$. We can easily find a set of differential equations for this flow. Setting