Chapter 2
Manifolds, Vectorfields, Lie Brackets, Distributions

In the previous chapter we have seen that many nonlinear control systems

\[
\begin{align*}
\dot{x} &= f(x, u) \\
y &= h(x, u)
\end{align*}
\]  

(2.1)

are properly defined on a state space which is not equal to Euclidean space \( \mathbb{R}^n \), but instead is a curved \( n \)-dimensional subset of \( \mathbb{R}^m \) for some \( m \), called a manifold. As a consequence, the equations (2.1) usually do not describe the system everywhere, but only on a part of the state space, and for a different part of the state space we generally need another representation of the system in equations like (2.1). In geometric language we say that (2.1) is a local coordinate expression of the system we wish to describe, and that in order to cover the whole system more than one coordinate expression is needed. For patching together these different expressions the notion of coordinate transformation is instrumental.

In the first part of this chapter (Section 2.1) we will develop the mathematical machinery to do this in a proper way. The approach taken here enables us to define a nonlinear control system on a curved state space independently of any choice of local coordinates. This so-called coordinate-free viewpoint is often illuminating, and can provide shortcuts in calculations which may be very tedious in local coordinates. The material covered in Section 2.1 is not easy to grasp at first reading. On the other hand, in order to be able to read Section 2.2 and the next chapters it is not really necessary to understand this material in full detail. In fact for most of the chapters to follow a rough understanding of the main ideas will be sufficient. Therefore we give before Section 2.1 a rough and intuitive survey of the material covered. One is advised first to read this survey and to pass swiftly over Section 2.1, and then to re-read Section 2.1 at later occasions.

For the second part of the chapter (Section 2.2) we will follow a different path. Since this material is much more at the heart of the contents of this book we will first go through it in reasonable detail, and give a short summary of the material afterwards (Section 2.3).
In Section 2.2 we will dwell more specifically upon the properties of sets of ordinary differential equations (without inputs) defined on a manifold which in a coordinate-free approach are described as \textit{vectorfields}. We show how under certain conditions there exist proper choices of local coordinates in which the vectorfield or the collection of vectorfields take an easy form, which is very much amenable for further analysis. These tools will prove to be instrumental for much of the theory developed in the chapters that will follow.

\section{2.0 Survey of Section 2.1}

\textit{Manifold}

Consider a topological space \(M\) (i.e., we know what subsets \(U \subset M\) are called open). Suppose that for any \(p \in M\) there exists an open set \(U\) containing \(p\), and a bijection \(\varphi\) mapping \(U\) onto some open subset of \(\mathbb{R}^n\) for some fixed \(n\). On \(\mathbb{R}^n\) we have the natural coordinate functions \(r_i(a_1, \ldots, a_n) = a_i, i \in \mathbb{n} := (1, \ldots, n)\). By composition with \(\varphi\) we obtain coordinate functions \(x_i, i \in \mathbb{n},\) on \(U\) by letting

\[ x_i = r_i \circ \varphi, \quad i \in \mathbb{n}. \tag{2.2} \]

In this way the grid defined on \(\varphi(U) \subset \mathbb{R}^n\) by the coordinate functions \(r_i, i \in \mathbb{n}\), transforms into a grid on \(U \subset M\).

The open set \(U\) together with the map \(\varphi\) is called a \textit{coordinate chart} and is also denoted as \((U, x_1, \ldots, x_n)\). On \(\mathbb{R}^n\) there is a natural notion of differentiability, and we would like to transfer this notion to \(M\), so that we can talk about differentiable functions on \(M\). In order to do this we have to impose extra conditions on the coordinate charts \((U, \varphi)\) as above. First of all we require that for any chart \((U, \varphi)\) the map \(\varphi\) is a \textit{homeomorphism} (\(\varphi\) and \(\varphi^{-1}\) are continuous). Secondly we require that all charts are \((C^\infty)-\)compatible. This means that for any two charts \((U, \varphi), (V, \psi)\) with \(U \cap V \neq \emptyset\) the map

\[ \varphi \circ \psi^{-1} \]