Tangent Cones

The Zariski tangent space to a variety $X \subset \mathbb{A}^n$ at a point $p$ is described by taking the linear part of the expansion around $p$ of all the functions on $\mathbb{A}^n$ vanishing on $X$. In case $p$ is a singular point of $X$, however, this does not give us a very refined picture of the local geometry of $X$; for example, if $X \subset \mathbb{A}^2$ is a plane curve, the Zariski tangent space to $X$ at any singular point $p$ will be all of $T_p(\mathbb{A}^2) = K^2$. We will describe here the tangent cone, an object that, while it certainly does not give a complete description of the local structure of a variety at a singular point, is at least a partial refinement of the notion of tangent space.

The definition of the tangent cone is simple enough. In the definition of the tangent space of a variety $X \subset \mathbb{A}^n$ at a point $p$ we took all $f \in I(X)$, expanded around $p$ and took their linear parts; we defined $T_p(X)$ to be the zero locus of these homogeneous linear forms. The difference in the definition of the tangent cone $TC_p(X)$ to $X$ at $p$ is this: we take all $f \in I(X)$, expand around $p$, and look not at their linear terms but at their leading terms (that is, their terms of lowest degree), whatever the degree of those terms might be. We then define the tangent cone $TC_p(X)$ to be the subvariety of $\mathbb{A}^n$ defined by these leading terms. As we have defined it, the tangent cone is a subvariety of the ambient space $\mathbb{A}^n$; but since the linear forms defining $T_p(X)$ are among the leading terms of $F \in I(X)$, it will turn out to be a subvariety of the Zariski tangent space $T_p(X)$, via the inclusion of $T_p(X)$ in the tangent space $T_p(\mathbb{A}^n) = \mathbb{A}^n$.

In the simplest case, that of a hypersurface $X \subset \mathbb{A}^n$ given by one polynomial $f(x_1, \ldots, x_n)$ and $p = (0, \ldots, 0)$, we write

$$f(x) = f_m(x) + f_{m+1}(x) + \cdots$$

where $f_k(x)$ is homogeneous of degree $k$ in $x_1, \ldots, x_n$; the tangent cone will be the cone of degree $m$ given by the homogeneous polynomial $f_m$. Thus, for example,
the tangent cone $TC_p(X)$ to a plane curve $X \subset \mathbb{A}^2$ with a node at $p$ will be the union of the two lines tangent to its branches at $p$, while the tangent cone to a curve with a cusp will be a single line.

(Note that since the linear term of the sum of two power series is the sum of their linear terms, in defining the Zariski tangent space to a variety $X \subset \mathbb{A}^n$ we needed to look only at the linear parts of a set of generators of the ideal $I(X)$. By contrast, the leading term of a sum will not always be in the ideal generated by their leading terms; to describe the tangent cone in general we will have to take the leading terms of all $f \in I(X)$.)

After giving the initial definition of the Zariski tangent space to an embedded variety $X \subset \mathbb{A}^n$, we gave a more visibly intrinsic definition in terms of the ring of regular functions on $X$; we can do likewise for the tangent cone. Specifically, let $\mathcal{O} = \mathcal{O}_{X, p}$ be the ring of germs of regular functions on $X$ at $p$ and $m \subset \mathcal{O}$ the maximal ideal in this local ring. $\mathcal{O}$ has a filtration by powers of the ideal $m$:

$$\mathcal{O} \supset m \supset m^2 \supset m^3 \supset \cdots$$

and we define the ring $B$ to be the associated graded ring, that is,

$$B = \bigoplus_{a=0}^{\infty} m^a / m^{a+1}.$$ 

This is by definition generated by its first graded piece

$$B_1 = m / m^2$$

and so $B$ is a quotient ring $A/I$ of the symmetric algebra

$$A = \bigoplus_{a=0}^{\infty} \text{Sym}^a(m/m^2)$$

$$= \bigoplus_{a=0}^{\infty} \text{Sym}^a((T_p X)*) .$$

Now, $A$ is naturally the ring of regular functions on the Zariski tangent space $T_p(X) = m / m^2$; we may define the tangent cone $TC_p(X)$ to be the subvariety of $T_p(X)$ defined by this quotient, that is, the common zero locus of the polynomials $g \in I \subset A$. 

\[ X : (y^2 - x^2(x+1)) \]

\[ X : (y^2 - x^3) \]

\[ TC_p(X) : (y^2 - x^2) \]

\[ TC_p(X) : (y^2) \]