Example 3.1. Cones

We start here with a hyperplane \( \mathbb{P}^{n-1} \subset \mathbb{P}^n \) and a point \( p \in \mathbb{P}^n \) not lying on \( \mathbb{P}^{n-1} \); if we like, we can take coordinates \( Z \) on \( \mathbb{P}^n \) so that \( \mathbb{P}^{n-1} \) is given by \( Z_n = 0 \) and the point \( p = [0, \ldots, 0, 1] \). Let \( X \subset \mathbb{P}^{n-1} \) be any variety. We then define the cone \( \overline{X, p} \) over \( X \) with vertex \( p \) to be the union

\[
\overline{X, p} = \bigcup_{q \in X} \overline{qp}
\]

of the lines joining \( p \) to points of \( X \). (If \( p \) lies on the hyperplane at infinity, \( X, p \) will look like a cylinder rather than a cone; in projective space these are the same thing.) \( \overline{X, p} \) is easily seen to be a variety: if we choose coordinates as earlier and \( X \subset \mathbb{P}^{n-1} \) is the locus of polynomials \( F_a = F_a(Z_0, \ldots, Z_{n-1}) \), the cone \( \overline{X, p} \) will be the locus of the same polynomials \( F_a \) viewed as polynomials in \( Z_0, \ldots, Z_n \).

As a slight generalization of the cone construction, let \( \Lambda \cong \mathbb{P}^k \subset \mathbb{P}^n \) and \( \Psi \cong \mathbb{P}^{n-k-1} \) be complementary linear subspaces (i.e., disjoint and spanning all of \( \mathbb{P}^n \)), and let \( X \subset \Psi \) be any variety. We can then define the cone \( \overline{X, \Lambda} \) over \( X \) with vertex \( \Lambda \) to be the union of the \( (k+1) \)-planes \( \overline{q, \Lambda} \) spanned by \( \Lambda \) together with points \( q \in X \). Of course, this construction represents merely an iteration of the preceding one; we can also construct the cone \( \overline{X, \Lambda} \) by taking the cone over \( X \) with vertex a point \( k + 1 \) times.
Exercise 3.2. Let $\Psi$ and $\Lambda \subset \mathbb{P}^n$ be complementary linear subspaces as earlier, and $X \subset \Psi$ and $Y \subset \Lambda$ subvarieties. Show that the union of all lines joining points of $X$ to points of $Y$ is a variety.

In Lecture 8, we will see an analogous way of constructing a variety $\overline{X, Y}$ for any pair of varieties $X, Y \subset \mathbb{P}^n$.

Example 3.3. Quadrics

We can use the concept of cone to give a uniform description of quadric hypersurfaces, at least in case the characteristic of the field $K$ is not 2. To begin with, a quadric hypersurface $Q \subset \mathbb{P}V = \mathbb{P}^n$ is given as the zero locus of a homogeneous quadratic polynomial $Q: V \to K$. Now assume that $\text{char}(K) \neq 2$. The polynomial $Q$ may be thought of as the quadratic form associated to a bilinear form $Q_0$ on $V$, that is, we may write

$$Q(v) = Q_0(v, v),$$

where $Q_0: V \times V \to K$ is defined by

$$Q_0(v, w) = \frac{Q(v + w) - Q(v) - Q(w)}{2}.$$

Note that $Q_0$ is both symmetric and bilinear. There is also associated to $Q_0$ the corresponding linear map

$$\overline{Q}: V \to V^*$$

given by sending $v$ to the linear form $Q(v, \cdot)$, i.e., by setting

$$\overline{Q}(v)(w) = \overline{Q}(w)(v) = Q_0(v, w).$$

Now, to classify quadrics, note that any quadric $Q$ on a vector space $V$ may be written, in terms of a suitably chosen basis, as

$$Q(X) = X_0^2 + X_1^2 + \cdots + X_r^2.$$

To see this, we choose the basis $e_0, \ldots, e_n$ for $V$ as follows. First, we choose $e_0$ such that $Q(e_0) = 1$; then we choose $e_1 \in (K e_0)^\perp$ (i.e., such that $Q_0(e_0, e_1) = 0$) such that $Q(e_1) = 1$, and so on, until $Q$ vanishes identically on $(K e_0 + \cdots + K e_k)^\perp$. Finally, we may complete this to a basis with an arbitrary basis $e_{k+1}, \ldots, e_n$ for $(K e_0 + \cdots + K e_k)^\perp$. We say in this case that the quadric $Q$ has rank $k + 1$; note that $k + 1$ is also the rank of the linear map $\overline{Q}$. By this, a quadric is determined up to projective motion by its rank.

Note that as in Example 1.20, we are led to define a quadric hypersurface in general to be an equivalence class of nonzero homogeneous quadratic polynomials; two such polynomials are equivalent if they differ by multiplication by a scalar. The one additional object that this introduces into the class of quadrics is the double plane, that is, the quadric associated to the square $Q = L^2$ of a linear polynomial $L$. 