LECTURE 5

Ideals of Varieties, Irreducible
Decomposition, and the Nullstellensatz

Generating Ideals

The time has come to talk about the various senses in which a variety may be
defined by a set of equations. There are three different meanings of the statement
that a collection of polynomials \( \{ F_a(Z) \} \) "cut out" a variety \( X \subset \mathbb{P}^n \), and several
different terms are used to convey each of these meanings.

Let's start with the affine case, where there are only two possibilities. Let \( X \subset \mathbb{A}^n \)
be a variety and \( \{ f_a(z_1, \ldots, z_n) \}_{a=1, \ldots, m} \) a collection of polynomials. When we say
that the polynomials \( f_a \) determine \( X \), we could a priori mean one of two things:

(i) the common zero locus \( V(f_1, \ldots, f_m) \) of the polynomials \( f_a \) is \( X \) or
(ii) the polynomials \( f_a \) generate the ideal \( \mathfrak{I}(X) \).

Clearly, the second is stronger. For example, the zero locus of the polynomial
\( x^2 \in K[x] \) is the origin \( 0 \in \mathbb{A}^1 \), but the ideal of functions vanishing at 0 is \( (x) \),
not \( (x^2) \). In general, the ideal of functions vanishing on a variety has the property
that, for any polynomial \( f \in K[z_1, \ldots, z_n] \), if a power \( f^k \in I \) then \( f \in I \). We formalize
this by observing that for any ideal \( I \) in a ring \( R \), the set of all elements \( f \in R \)
such that \( f^k \in I \) for some \( k > 0 \) is again an ideal, called the radical of \( I \) and denoted \( \mathfrak{r}(I) \). We call an ideal \( I \) radical if it is equal to \( \mathfrak{r}(I) \); as we have just observed, an ideal
without this property cannot be of the form \( \mathfrak{I}(X) \).

To put it another way, we have a two-way correspondence

\[
\begin{array}{ccc}
\{ \text{subvarieties} \} & \overset{I}{\longrightarrow} & \{ \text{ideals} \} \\
\text{of } \mathbb{A}^n & \leftarrow & \{ I \subset K[z_1, \ldots, z_n] \}
\end{array}
\]

but this is not by any means bijective: in one direction, the composition of the
two is the identity—the definition of a variety \( X \subset \mathbb{A}^n \) amounts to the statement

J. Harris, Algebraic Geometry
that $V(I(X)) = X$—but going the other way the composition is neither injective nor surjective. We can fix this by simply restricting our attention to the image of the map $I$, and happily there is a nice characterization of this image (and indeed of the composition $I \circ V$). This is the famous Nullstellensatz:

**Theorem 5.1.** For any ideal $I \subseteq K[z_1, \ldots, z_n]$, the ideal of functions vanishing on the common zero locus of $I$ is the radical of $I$, i.e.,

$$I(V(I)) = r(I)$$

Thus, there is a bijective correspondence between subvarieties $X \subseteq \mathbb{A}^n$ and radical ideals $I \subseteq K[z_1, \ldots, z_n]$.

We will defer both the proof of the Nullstellensatz and some of its corollaries to later in this lecture and will proceed with our discussion now.

Note that, as one consequence of the Nullstellensatz, we can say that a $K$-algebra $A$ occurs as the coordinate ring of an affine variety if and only if $A$ is finitely generated and has no nilpotents. Clearly these two conditions are necessary; if they are satisfied, we can write

$$A = K[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

so that we will have $A = A(X)$, where $X \subseteq \mathbb{A}^n$ is the zero locus of the polynomials $f_i$.

At this point we can take a minute out and mention one of the fundamental notions of scheme theory. Basically, if one is going to fix the correspondence on page 48 so as to make it bijective, there are naively two ways of going about it. We can either restrict the class of objects on the right or enlarge the class of objects on the left. In classical algebraic geometry, as we have just said, we do the former; in scheme theory, we do the latter. Thus, we define an affine scheme $X \subseteq \mathbb{A}^n$ to be an object associated to an arbitrary ideal $I \subseteq K[z_1, \ldots, z_n]$.

What sense can this possibly make? This is not the place to go into it in any detail, but we may remark that, in fact, most of the notions that we actually deal with in algebraic geometry are defined in terms of rings and ideals as well as in terms of subsets of affine or projective spaces. For example, if $X \subseteq \mathbb{A}^n$ is a variety with ideal $I = I(X)$, we define a function on $X$ to be an element of the ring $A(X) = K[z_1, \ldots, z_n]/I$; the intersection of two such varieties $X, Y \subseteq \mathbb{A}^n$ is given by the join of their ideals; the data of a map between two such varieties $X$ and $Y$ are equivalent to the data of a map $A(Y) \rightarrow A(X)$, and so on. The point is that all these things make sense whether or not $I$ is a radical ideal. The scheme associated to an arbitrary ideal $I \subseteq K[z_1, \ldots, z_n]$ may not seem like a geometric object, especially in case $I$ is not radical, but it does behave formally like one and it encodes extra information that is of geometric interest.

Before going on, we will introduce some terminology. We say that a collection $\{f_a\}$ of polynomials cut out a variety $X \subseteq \mathbb{A}^n$ set-theoretically to mean just that their common zero locus $V(\{f_a\}) = X$; we say that they cut out $X$ scheme-theoretically, or ideal-theoretically, if in fact they generate the ideal $I(X)$.