Chapter 4  
Applications of Derivatives

Section 1. Monotonicity

Webster’s New World Dictionary, Third Collegiate Edition, gives five meanings to “monotone”, all musical, none mathematical. In math, a function is monotone on an interval if it is either increasing or decreasing on the interval.

![Increasing Functions](image1.png)  
![Decreasing Functions](image2.png)

Figure 1.1. Increasing Functions  
Figure 1.2. Decreasing Functions

**Definition—Monotone Increasing/Decreasing:** Let $F(x)$ be defined on interval $I$.

1. $F(x)$ is (monotone) increasing on $I$ if $F(x_1) \leq F(x_2)$ whenever $x_1 < x_2$.
2. $F(x)$ is strictly increasing on $I$ if $F(x_1) < F(x_2)$ whenever $x_1 < x_2$.
3. $F(x)$ is (monotone) decreasing on $I$ if $F(x_1) \geq F(x_2)$ whenever $x_1 < x_2$.
4. $F(x)$ is strictly decreasing on $I$ if $F(x_1) > F(x_2)$ whenever $x_1 < x_2$.

(It is understood that $x_1$ and $x_2$ are points of the interval $I$.)
Example 1
Graph \( y = F(x) = x^3 \). This function is defined on the reals \( \mathbb{R} \). It is clear from the graph that \( F(x) \) is strictly increasing. It is also relatively easy to prove it so by elementary methods in this particular example. However, with an eye to more complicated functions, note that the derivative

\[
F'(x) = 3x^2 \geq 0 \quad (=0 \text{ only for } x = 0)
\]

Of course, this should be no surprise. The derivative is the rate of change of the function. A positive derivative should mean that the function is increasing; a negative derivative should mean that the function is decreasing. □

Theorem—Derivative Test
Let \( F(x) \) be defined and differentiable on \( I \).
1. If \( F'(x) \geq 0 \) at each point of \( I \), then \( F(x) \) is increasing.
2. If \( F'(x) > 0 \) at each point of \( I \), then \( F(x) \) is strictly increasing.
3. If \( F'(x) \leq 0 \) at each point of \( I \), then \( F(x) \) is decreasing.
4. If \( F'(x) < 0 \) at each point of \( I \), then \( F(x) \) is strictly decreasing.

Each of these statements should seem obvious. Yet, when you try to prove them, you seem to be up against a wall. The next result shows what is going on.

Example 2
Take \( F(x) = x^3 - 3x \). Differentiate:

\[
F'(x) = 3x^2 - 3 = 3(x^2 - 1)
\]

Then \( F'(x) > 0 \) provided \( x^2 > 1 \), that is, provided \( x < -1 \) or \( x > 1 \). Similarly \( F'(x) < 0 \) provided \( x^2 < 1 \), that is, \( -1 < x < 1 \).

We conclude that

\[
F(x) \begin{cases} 
\text{increases} & \text{on } (-\infty, -1) \\
\text{decreases} & \text{on } (-1, 1) \\
\text{increases} & \text{on } (1, \infty)
\end{cases}
\]

The graph (Figure 1.3) of \( y = x^3 - 3x \) confirms this. Note for future work the situations at \( x = -1 \) and at \( x = 1 \). □