1.5
Spaces of Bounded Functions

Let $T$ be an arbitrary set. The space $\ell^\infty(T)$ is defined as the set of all uniformly bounded, real functions on $T$: all functions $z: T \mapsto \mathbb{R}$ such that

$$\|z\|_T := \sup_{t \in T} |z(t)| < \infty.$$ 

It is a metric space with respect to the uniform distance $d(z_1, z_2) = \|z_1 - z_2\|_T$.

The space $\ell^\infty(T)$, or a suitable subspace of it, is a natural space for stochastic processes with bounded sample paths. A stochastic process is simply an indexed collection $\{X(t): t \in T\}$ of random variables defined on the same probability space: every $X(t): \Omega \mapsto \mathbb{R}$ is a measurable map. If every sample path $t \mapsto X(t, \omega)$ is bounded, then a stochastic process yields a map $X: \Omega \mapsto \ell^\infty(T)$. Sometimes the sample paths have additional properties, such as measurability or continuity, and it may be fruitful to consider $X$ as a map into a subspace of $\ell^\infty(T)$. If in either case the uniform metric is used, this does not make a difference for weak convergence of a net; but for measurability it can. Here is one example of this situation; more examples are discussed in the next section.

1.5.1 Example (Continuous functions). Let $T$ be a compact semimetric space; for instance, a compact interval in the real line, or the extended real line $[-\infty, \infty]$ with the metric $\rho(s, t) = |\arctan s - \arctan t|$. The set $C(T)$ of all continuous functions $z: T \mapsto \mathbb{R}$ is a separable, complete subspace of $\ell^\infty(T)$. The Borel $\sigma$-field of $C(T)$ equals the $\sigma$-field generated by the coordinate projections $z \mapsto z(t)$, the projection $\sigma$-field (Problem 1.7.1).

A. W. van der Vaart et al., Weak Convergence and Empirical Processes
Thus a map $X: \Omega \mapsto C(T)$ is Borel measurable if and only if it is a stochastic process.

In most cases a map $X: \Omega \mapsto \ell^\infty(T)$ is a stochastic process. The small amount of measurability this gives may already be enough for asymptotic measurability. The special role played by the marginals $(X(t_1), \ldots, X(t_k))$, which are considered as maps into $\mathbb{R}^k$, is underlined by the following three results. Weak convergence in $\ell^\infty(T)$ can be characterized as asymptotic tightness plus convergence of marginals.

1.5.2 Lemma. Let $X_\alpha: \Omega_\alpha \mapsto \ell^\infty(T)$ be asymptotically tight. Then it is asymptotically measurable if and only if $X_\alpha(t)$ is asymptotically measurable for every $t \in T$.

1.5.3 Lemma. Let $X$ and $Y$ be tight Borel measurable maps into $\ell^\infty(T)$. Then $X$ and $Y$ are equal in Borel law if and only if all corresponding marginals of $X$ and $Y$ are equal in law.

1.5.4 Theorem. Let $X_\alpha: \Omega_\alpha \mapsto \ell^\infty(T)$ be arbitrary. Then $X_\alpha$ converges weakly to a tight limit if and only if $X_\alpha$ is asymptotically tight and the marginals $(X_\alpha(t_1), \ldots, X_\alpha(t_k))$ converge weakly to a limit for every finite subset $t_1, \ldots, t_k$ of $T$. If $X_\alpha$ is asymptotically tight and its marginals converge weakly to the marginals $(X(t_1), \ldots, X(t_k))$ of a stochastic process $X$, then there is a version of $X$ with uniformly bounded sample paths and $X_\alpha \Rightarrow X$.

Proofs. For the proof of both lemmas, consider the collection $\mathcal{F}$ of all functions $f: \ell^\infty(T) \mapsto \mathbb{R}$ of the form

$$f(z) = g(z(t_1), \ldots, z(t_k)), \quad g \in C_b(\mathbb{R}^k), \ t_1, \ldots, t_k \in T, \ k \in \mathbb{N}.$$ 

This forms an algebra and a vector lattice, contains the constant functions, and separates points of $\ell^\infty(T)$. Therefore, the lemmas are corollaries of Lemmas 1.3.13 and 1.3.12, respectively.

If $X_\alpha$ is asymptotically tight and the marginals converge, then $X_\alpha$ is asymptotically measurable by the first lemma. By Prohorov's theorem, $X_\alpha$ is relatively compact. To prove weak convergence, it suffices to show that all limit points are the same. This follows from marginal convergence and the second lemma.

Marginal convergence can be established by any of the well-known methods for proving weak convergence on Euclidean space. Tightness can be given a more concrete form, either through finite approximation or (essentially) with the help of the Arzelà-Ascoli theorem. Finite approximation leads to the simper of the two characterizations, but the second approach