15

Normal-Ogive Multidimensional Model

Roderick P. McDonald

Introduction

In an attempt to provide a unified foundation for common factor analysis, true score theory, and latent trait (item response) theory, McDonald (1962a, 1962b, 1967) defined a general strong principle of local independence and described a general latent trait model, as follows: Let $\mathbf{U}$ be a $n \times 1$ random vector of manifest variables—test or possibly binary item scores—and θ a $k \times 1$ random vector of latent traits—not yet defined. The strong principle of local independence, which defines θ and the dimension k of the vector $\mathbf{U}$, states that

$$g\{\mathbf{U} \mid \theta\} = \prod_{i=1}^{k} g_i\{U_i \mid \theta\} \tag{1}$$

where $g\{\ \}$ is the conditional density of $\mathbf{U}$ and $g_i\{\ \}$ is the conditional density of the ith component. (Note that θ is not necessarily continuous and may consist of a dummy variable defining a latent class model.)

The strong principle of local independence implies the weak principle, which may be defined by writing

$$E\{\mathbf{U} \mid \theta\} = \phi(\theta) \tag{2}$$

where ϕ is a vector of functions, defining a vector of residuals

$$\mathbf{e} = \mathbf{U} - E\{\mathbf{U} \mid \theta\} \tag{3}$$

(whence, axiomatically, $\mathrm{Cov}\{\phi, \mathbf{e}\} = \mathbf{0}$) and

$$\mathrm{Cov}\{\mathbf{e}\} = \Delta$$

diagonal, positive definite. It follows without further assumptions that the covariance structure of $\mathbf{U}$ is given by

$$\mathrm{Cov}\{\mathbf{U}\} = \sum = \mathrm{Cov}\{\phi\} + \Delta. \tag{4}$$

W. J. van der Linden et al. (eds.), *Handbook of Modern Item Response Theory*

The k components of θ account for all dependencies in probability of $U_1, \ldots, U_n$, under the strong principle of local independence, and for all their covariances under the weak principle. This fact justifies speaking of $\theta_1, \ldots, \theta_k$ indifferently as latent traits or common factors, and interpreting them in applications as those traits or states of the examinees that $U_1, \ldots, U_n$ serve to indicate or, we may say, define. In applications of latent trait theory to cognitive tests or items it is reasonable to refer to the latent traits as "abilities." Especially in multidimensional IRT it is very important to recognize that they are equally applicable to any social science data—e.g., personality inventories and attitude scales. (The problematic identification of a latent trait as a "true score" and the residuals as "errors of measurement" must rest upon extra mathematical considerations, if not on mere habits of thought.) From the manner in which latent traits are interpreted, it seems reasonable to say that when a structural model is fitted to data and tested on the basis of univariate and bivariate information only, and thus invoking only the weak form of the principle of local independence, it is unlikely that the investigator seriously imagines that the conditional covariances vanish while the variables still possess higher-order mutual statistical dependencies. Of course such higher-order dependencies are logically possible, and would account for any systematic differences between results obtained from bivariate information and results from the full information in the data.

McDonald (1982) suggested a fundamental classification of the family of latent trait models into: (1) *strictly linear models* in which the functions $\phi(\theta)$ are linear both in the coefficients of the regressions on the latent traits and in the latent traits themselves; (2) *wide-sense linear models* in which the functions are linear in the coefficients but not in the latent traits; and (3) *strictly nonlinear models* which cannot be expressed as a wide-sense linear model with a finite number of terms. The linear common factor model and the latent class model are examples of a strictly linear model. The polynomial factor model (McDonald, 1967; Etezadi-Amoli and McDonald, 1983) is wide-sense linear. The logistic and normal-ogive models for binary data are strictly nonlinear.

A central result of the early work on the general latent trait model is the demonstration (McDonald, 1967) that if a wide-sense linear model contains r functions of the k latent traits, then the covariance structure of the manifest variables can be expressed as

$$\sum = \Lambda\Lambda' + \Delta \tag{5}$$

where Λ is a $n \times r$ matrix, and Δ is as defined above. It follows that such a model cannot be distinguished from a strictly linear latent trait model with r latent traits on the basis of bivariate information alone. Exploratory methods were developed to enable such distinctions on the basis of the distribution of the latent traits in the latent space. The work described