Sets and Mappings

In this chapter, we have put together a number of definitions concerning the basic terminology of mathematics. The reader could reasonably start reading Chapter I immediately, and refer to the present chapter only when he comes across a word which he does not understand. Most concepts will in fact be familiar to most readers.

We shall use some examples which logically belong with later topics in the book, but which most readers will have already encountered. Such examples are needed to make the text intelligible.

0, §1. SETS

A collection of objects is called a set. A member of this collection is also called an element of the set. If \( a \) is an element of a set \( S \), we also say that \( a \) lies in \( S \), and write \( a \in S \). To denote the fact that \( S \) consists of elements \( a, b, \ldots \) we often use the notation \( S = \{a, b, \ldots\} \). We assume that the reader is acquainted with the set of positive integers, denoted by \( \mathbb{Z}^+ \), and consisting of the numbers 1, 2, \ldots. The set consisting of all positive integers and the number 0 is called the set of natural numbers. It is denoted by \( \mathbb{N} \).

A set is often determined by describing the properties which an object must satisfy in order to be in the set. For instance, we may define a set \( S \) by saying that it is the set of all real numbers \( \geq 1 \). Sometimes, when defining a set by certain conditions on its elements, it may happen that there is no element satisfying these conditions. Then we say that the set is empty. Example: The set of all real numbers \( x \) which are both \( > 1 \) and \( < 0 \) is empty because there is no such number.
If $S$ and $S'$ are sets, and if every element of $S'$ is an element of $S$, then we say that $S'$ is a subset of $S$. Thus the set of all even positive integers $\{2, 4, 6, \ldots\}$ is a subset of the set of positive integers. To say that $S'$ is a subset of $S$, but $S' \neq S$, then we shall say that $S'$ is a proper subset of $S$. Thus the set of even integers is a proper subset of the set of natural numbers. To denote the fact that $S'$ is a subset of $S$, we write $S' \subset S$, or $S \supset S'$; we also say that $S'$ is contained in $S$.

If $S' \subset S$ and $S \subset S'$ then $S = S'$.

If $S_1$, $S_2$ are sets, then the intersection of $S_1$ and $S_2$, denoted by $S_1 \cap S_2$, is the set of elements which lie in both $S_1$ and $S_2$. For instance, if $S_1$ is the set of natural numbers $\geq 3$, and $S_2$ is the set of natural numbers $\leq 3$, then $S_1 \cap S_2 = \{3\}$ is the set consisting of the number 3 alone.

The union of $S_1$ and $S_2$, denoted by $S_1 \cup S_2$, is the set of elements which lie in $S_1$ or $S_2$. For example, if $S_1$ is the set of all odd numbers $\{1, 3, 5, 7, \ldots\}$ and $S_2$ consists of all even numbers $\{2, 4, 6, \ldots\}$, then $S_1 \cup S_2$ is the set of all positive integers.

Finally, if $S$, $T$ are sets, we denote by $S \times T$ the set of all pairs $(x, y)$ with $x \in S$ and $y \in T$. Note that if $S$ or $T$ is empty, then $S \times T$ is also empty. Similarly, if $S_1, \ldots, S_n$ are sets, we denote by $S_1 \times \cdots \times S_n$, or

$$
\prod_{i=1}^{n} S_i
$$

the set of all $n$-tuples $(x_1, \ldots, x_n)$ with $x_i \in S_i$.

0, §2. MAPPINGS

Let $S$, $T$ be sets. A mapping or map, from $S$ to $T$ is an association which to every element of $S$ associates an element of $T$. Instead of saying that $f$ is a mapping of $S$ into $T$, we shall often write the symbols $f: S \to T$.

If $f: S \to T$ is a mapping, and $x$ is an element of $S$, then we denote by $f(x)$ the element of $T$ associated to $x$ by $f$. We call $f(x)$ the value of $f$ at $x$, or also the image of $x$ under $f$. The set of all elements $f(x)$, for all $x \in S$, is called the image of $f$. If $S'$ is a subset of $S$, then the set of elements $f(x)$ for all $x \in S'$, is called the image of $S'$ and is denoted by $f(S')$.

If $f$ is as above, we often write $x \mapsto f(x)$ to denote the association of $f(x)$ to $x$. We thus distinguish two types of arrows, namely $\to$ and $\mapsto$. 