The Fourier Integral

XIV, §1. THE SCHWARTZ SPACE

We are going to define a space of functions such that any operation we want to make on improper integrals converges for functions in that space.

Let $f$ be a continuous function on $\mathbb{R}$. We say that $f$ is rapidly decreasing at infinity if for every integer $m > 0$ the function $|x|^m f(x)$ is bounded. Since $|x|^{m+1} f(x)$ is bounded, it follows that

$$\lim_{|x| \to \infty} |x|^m f(x) = 0$$

for every positive integer $m$.

We let $S$ be the set of all infinitely differentiable functions $f$ such that $f$ and every one of its derivatives decrease rapidly at infinity. There are such functions, for instance $e^{-x^2}$.

It is clear that $S$ is a vector space over $\mathbb{C}$. (We take all functions to be complex valued.) Every function in $S$ is bounded. If $f \in S$, then its derivative $Df$ is also in $S$, and hence so is the $p$-th derivative $D^p f$ for every integer $p \geq 0$. We call $S$ the Schwartz space. Since

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx$$

converges, it follows that every function in $S$ can be integrated over $\mathbb{R}$, i.e. the integral

$$\int_{-\infty}^{\infty} f(x) \, dx$$
converges absolutely. For simplicity, from now on we write

$$\int = \int_{-\infty}^{\infty}$$

since we don't deal with any other integrals.

If $P$ is a polynomial, say of degree $m$, then there is a number $C > 0$ such that for all $|x|$ sufficiently large, we have

$$|P(x)| \leq C|x|^m.$$ 

Hence if $f \in S$, then $Pf$ also lies in $S$. If $f, g \in S$ then $fg \in S$. (Obvious.) We see that $S$ is an algebra under ordinary multiplication of functions.

We shall have to consider the function $-ixf(x)$, i.e. multiply by $-ix$. To avoid the $x$, we may use the notation

$$(Mf)(x) = -ixf(x),$$

and iterate,

$$M^p f(x) = (-ix)^p f(x)$$

for every integer $p \geq 0$.

In order to preserve a certain symmetry in subsequent results, it is convenient to normalize integrals over $\mathbb{R}$ by multiplication by a constant factor, namely $1/\sqrt{2\pi}$. For this purpose, we introduce a notation. We write

$$\int f(x) \, dx_1 = \frac{1}{\sqrt{2\pi}} \int f(x) \, dx.$$ 

We now define the Fourier transform of a function $f \in S$ by the integral

$$\hat{f}(y) = \int f(x)e^{-ixy} \, dx_1.$$ 

The integral obviously converges absolutely. But much more:

**Theorem 1.1.** If $f \in S$, then $\hat{f} \in S$. We have

$$D^p\hat{f} = (M^p f)^\wedge \quad \text{and} \quad (D^p f)^\wedge = (-1)^p M^p \hat{f}.$$