Theorem 4.2 of Chapter XV gave us a significant criterion for the existence of a potential function, but falls short of describing completely the nature of global obstructions for its existence if we know that the vector field is locally integrable. The present chapter deals systematically with the obstruction, which will be seen to depend on a single vector field. The same considerations are used in subsequent courses on complex analysis and Cauchy's theorem. The fundamental result proved in the present chapter is valid more generally, but will constitute perfect preparation for those who will subsequently deal with Cauchy's theorem. In fact, Emil Artin in the 1940s gave a proof of Cauchy's theorem basing the topological considerations (called homology) on the winding number (cf. his collected works). I have followed here Artin's idea, and applied it to locally integrable vector fields in an open set $U$ of $\mathbb{R}^2$. A fundamental result, quite independent of analysis, is that if the winding number of a closed rectangular path in $U$ is 0 with respect to every point outside $U$, then the path is a sum of boundaries of rectangles completely contained in $U$. See Theorem 3.2. If one knows that for certain vector fields their integrals around boundaries of rectangles are 0, then it immediately follows that their integrals along paths satisfying the above condition is also 0. This is the heart of the proof of the global integrability theorem, and may be viewed as a general theorem on circuits in the plane.

Some readers may be only interested in this aspect of locally integrable vector fields, and they may omit the entire subsequent discussion leading to homotopy. However, since I have found that homotopy is not satisfactorily treated from the point of view of an undergraduate analysis course anywhere else, I have still included three sections which deal with arbitrary continuous curves and homotopy.
In §1 we establish some technically convenient results about integrals along paths. In §2 we state the global integrability theorem, and mention some applications. We shall see that the vector field \( G \) mentioned in Chapter 15 is (up to translations) essentially the only obstruction for a locally integrable vector field to have a global potential function. In §3 we prove the global integrability theorem. In §4 we define the integral along an arbitrary continuous path. This is useful to deal with the homotopy form of the integrability theorem, which we give in §5. We discuss homotopies more extensively in §6.

**XVI, §1. ANOTHER DESCRIPTION OF THE INTEGRAL ALONG A PATH**

Let \( U \) be a connected open set in \( \mathbb{R}^2 \). Let \( F = (f_1, f_2) \) be a vector field on \( U \). For simplicity, we assume \( F \) is of class \( C^1 \). In §4 we shall indicate a generalization which allows more flexibility in dealing with certain questions. For our purposes here, we define \( F \) to be **locally integrable** if \( D_2 f_1 = D_1 f_2 \). Exactly the same proof given for Theorem 3.3 of Chapter XV shows that if \( D \) is a disc contained in \( U \), then \( F \) has a potential function on \( D \). All we needed was to be able to integrate along certain line segments from one point to another, and such integration is possible within a disc as well as within a rectangle.

By a **path** throughout until §4, we shall mean a piecewise \( C^1 \) path. If \( F \) has a potential function \( \varphi \) on \( U \) and \( \gamma \) is a path in \( U \) from a point \( P \) to a point \( Q \), then we know from Chapter XV that

\[
\int_{\gamma} F = \varphi(Q) - \varphi(P).
\]

Even if \( F \) does not admit a global potential function, it is still possible to express its integral locally in terms of such differences. We then extent the global formulation by using a partition as follows.

**Lemma 1.1.** Let \( \gamma: [a, b] \rightarrow U \) be a continuous curve in \( U \). Then there is some positive number \( r > 0 \) such that every point on the curve lies at distance \( \geq r \) from the complement of \( U \).

![Figure 1](image_url)