V, §1. CHARACTERIZATION OF THE INTEGRAL

The proof of the existence of the integral is best postponed until we have the language of normed vector spaces and uniform approximation. However, it is convenient to have the elementary integral available for examples, and exercises, and we need only know its properties, which can be conveniently summarized axiomatically.

**Theorem 1.1.** Let \( a, d \) be two real numbers with \( a < d \). Let \( f \) be a continuous function on \([a, d]\). Suppose that for each pair of numbers \( b \leq c \) in the interval we are able to associate a number denoted by \( I_b^c(f) \) satisfying the following properties:

1. If \( M, m \) are numbers such that \( m \leq f(x) \leq M \) for all \( x \) in the interval \([b, c]\), then
   \[
   m(c - b) \leq I_b^c(f) \leq M(c - b).
   \]
2. We have
   \[
   I_b^b(f) + I_b^c(f) = I_b^c(f).
   \]

Then the function \( x \mapsto I_a^x(f) \) is differentiable in the interval \([a, d]\), and its derivative is \( f(x) \).

**Proof.** We have the Newton quotient, say for \( h > 0 \),

\[
\frac{I_a^{x+h}(f) - I_a^x(f)}{h} = \frac{I_a^x(f) + I_a^{x+h}(f) - I_a^x(f)}{h} = \frac{I_a^{x+h}(f)}{h}.
\]
Let \( s \) be a point between \( x \) and \( x + h \) such that \( f \) reaches a minimum at \( s \) on the interval \([x, x + h]\), and let \( t \) be a point in this interval such that \( f \) reaches a maximum at \( t \). Let \( m = f(s) \) and \( M = f(t) \). Then by the first property,

\[
f(s)(x + h - x) \leq I_x^x f(x) \leq f(t)(x + h - x),
\]

whence

\[
f(s)h \leq I_x^x f(x) \leq f(t)h.
\]

Dividing by \( h \) shows that

\[
f(s) \leq \frac{I_x^x f(x)}{h} \leq f(t).
\]

As \( h \to 0 \), we see that \( s, t \to x \), and since \( f \) is continuous, by the squeezing process, we conclude that the limit of the Newton quotient exists and is equal to \( f(x) \).

If we take \( h < 0 \), then the argument proceeds entirely similarly. The Newton quotient is again squeezed between the maximum and the minimum values of \( f \) (there will be a double minus sign which makes it come out the same). We leave this part to the reader.

**Corollary 1.2.** An association as in Theorem 1.1 is uniquely determined. If \( F \) is any differentiable function on \([a, d]\) such that \( F' = f \), then

\[
I_a^x f(x) = F(x) - F(a).
\]

**Proof.** Both \( F \) and \( x \mapsto I_a^x f(x) \) have the same derivative, whence there is a constant \( C \) such that for all \( x \) we have

\[
F(x) = I_a^x f(x) + C.
\]

Putting \( x = a \) shows that \( C = F(a) \) and concludes the proof.

For convenience, we define

\[
I_a^b f(x) = -I_b^a f(x)
\]

whenever \( a \leq b \). Then property (2) is easily seen to be valid for any position of \( a, b, c \) in an interval on which \( f \) is continuous.