As demonstrated in the preceding chapters, the errors in most numerical solutions increase dramatically as the physical scale of the simulated disturbance approaches the minimum scale resolvable on the numerical mesh. When solving equations for which smooth initial data guarantees a smooth solution at all later times, such as the barotropic vorticity equation (3.123), any difficulties associated with poor numerical resolution can be avoided by using a sufficiently fine computational mesh. But if the governing equations allow an initially smooth field to develop shocks or discontinuities, as is the case with Burgers’s equation (3.113), there is no hope of maintaining adequate numerical resolution throughout the simulation, and special numerical techniques must be used to control the development of overshoots and undershoots in the vicinity of the shock. Numerical approximations to equations with discontinuous solutions must also satisfy additional conditions beyond the stability and consistency requirements discussed in Chapter 2 to guarantee that the numerical solution converges to the correct solution as the spatial grid interval and the time step approach zero.

The possibility of erroneous convergence to a function that does not approximate the true discontinuous solution can be demonstrated by comparing numerical solutions to the generalized Burgers’s equation in *advective form*

\[ \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial x} = 0 \]  

(5.1)

with those generated by analogous solutions to the same equation in *flux form*

\[ \frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\psi^3}{3} \right) = 0. \]  

(5.2)
FIGURE 5.1. Exact (dash-dotted), upstream advective-form (solid) and upstream flux-form (dashed) solutions to the generalized Burgers's equation at $t = 2.4$ on the subdomain $0.5 \leq x \leq 1$. (a) $\Delta x = 0.02, \Delta t = 0.01$; (b) $\Delta x = 0.005, \Delta t = 0.0025$.

As will be explained in Section 5.1, if the initial conditions are specified by the step function

$$
\psi(x, 0) = \begin{cases} 
1, & \text{if } x \leq 0, \\
0, & \text{otherwise},
\end{cases}
$$

(5.3)

the correct solution consists of a unit-amplitude step propagating to the right at speed $\frac{1}{3}$. An upstream finite-difference approximation to the advective form (5.1) was calculated using

$$
\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + \left( \frac{\phi_j^n + \phi_{j-1}^n}{2} \right)^2 \left( \frac{\phi_j^n - \phi_{j-1}^n}{\Delta x} \right) = 0,
$$

(5.4)

and an upstream approximation to the flux form (5.2) was obtained using

$$
\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + \frac{(\phi_j^n - \phi_{j-1}^n)^3}{3\Delta x} = 0.
$$

(5.5)

Figure 5.1a shows a comparison of the exact and the numerical solutions at $t = 2.4$ on the subdomain $0.5 \leq x \leq 1.0$. The computations were performed using $\Delta x = 0.02$ and a time step such that $\max(\psi(x, 0)) \Delta t / \Delta x = 0.5$. Both schemes yield plausible-looking approximations to the correct solution (shown by the thin dot-dashed line), but the numerical solution obtained using advective-form differencing (shown by the solid line) moves at the wrong speed. As illustrated in Fig. 5.1b, in which the numerical solutions are recalculated after reducing $\Delta x$ and $\Delta t$ by a factor of four, the speed of the solution generated by the advective-form approximation is not significantly improved by decreasing $\Delta x$ and $\Delta t$. The advective-form approximation simply does not converge to the correct solution in the limit $\Delta x \to 0$ and $\Delta t \to 0$. The difficulties that can be