Common Ancestry: The Nambu–Goto Action

The “hidden” symmetries and the associated transformation laws for the Chaplygin and Born–Infeld models may be given a coherent setting by considering the Nambu–Goto action for a d-brane in (d + 1) spatial dimensions, moving on (d + 1, 1)-dimensional space–time. In our context, a d-brane is simply a d-dimensional extended object; a 1-brane is a string, a 2-brane is a membrane, and so on. A d-brane in (d + 1) space divides that space in two.

The Nambu–Goto action reads

\[ I_{NG} = \int d\varphi^0 d\varphi \mathcal{L}_{NG} = \int d\varphi^0 d\varphi^1 \cdots d\varphi^d \sqrt{G}, \]  
\[ G = (-1)^d \det \frac{\partial X^\mu}{\partial \varphi^\alpha} \frac{\partial X^\mu}{\partial \varphi^\beta}. \]  

Here \( X^\mu \) is a (d+1, 1) target space–time (d-brane) variable, with \( \mu \) extending over the range \( \mu = 0,1,\ldots,d,d+1 \). The \( \varphi^\alpha \) are “world-volume” variables describing the extended object with \( \alpha \) ranging from \( \alpha = 0,1,\ldots,d \); \( \varphi^\alpha (\alpha = 1,\ldots,d) \) parametrizes the d-dimensional d-brane that evolves in \( \varphi^0 \).

The Nambu–Goto action is parametrization invariant, and we shall show that two different choices of parametrization (“light-cone” and “Cartesian”) lead to the Chaplygin gas and Born–Infeld actions, respectively. For both parametrizations we choose \((X^1,\ldots,X^d)\) to coincide with \((\varphi^1,\ldots,\varphi^d)\), renaming them as \( r \) (a d-dimensional vector). This is usually called the “static parametrization.” (The ability to carry out this parametrization globally presupposes that the extended object is topologically trivial; in the contrary situation, singularities will appear,
which are spurious in the sense that they disappear in different parametrizations, and parametrization-invariant quantities are singularity-free.)

4.1 Light-Cone Parameterization

For the light-cone parametrization we define \( X^\pm \) as \((X^0 \pm X^{d+1})/\sqrt{2}\). \( X^+ \) is renamed \( t \) and identified with \( \sqrt{2}\varphi^0 \). This completes the fixing of the parametrization and the remaining variable is \( X^- \), which is a function of \( \varphi^0 \) and \( \varphi \) or, after redefinitions, of \( t \) and \( r \). \( X^- \) is renamed as \( \theta(t,r) \) and then the Nambu–Goto action in this parametrization coincides with the Chaplygin gas action \( I_{\lambda}^{\text{Chaplygin}} \) in (3.26) [29].

4.2 Cartesian Parameterization

For the second, Cartesian parametrization \( X^0 \) is renamed \( ct \) and identified with \( c\varphi^0 \). The remaining target space variable \( X^{d+1} \), a function of \( \varphi^0 \) and \( \varphi \), equivalently of \( t \) and \( r \), is renamed \( \theta(t,r)/c \). Then the Nambu–Goto action reduces to the Born–Infeld action \( \int dt L_{a}^{\text{Born–Infeld}}, (3.46) \) [29].

4.3 Hodographic Transformation

There is another derivation of the Chaplygin gas from the Nambu–Goto action that makes use of a hodographic transformation, in which independent and dependent variables are interchanged. Although the derivation is more involved than the light-cone/static/parametrization used in Section 4.1, the hodographic approach is instructive in that it gives a natural definition for the density \( \rho \) which, in the above static parametrization approach, is determined from \( \theta \) by the Bernoulli equation (3.14).

We again use light-cone combinations: \((X^0 + X^{d+1})/\sqrt{2}\) is called \( \tau \) and is identified with \( \varphi^0 \), while \((X^0 - X^{d+1})/\sqrt{2}\) is