In the last section of Chapter 3 we introduced the Lebesgue $L^p$-spaces for general measures and discussed their most basic properties. The most important $L^p$-space, by far, is $L^2$. Its importance is its role in applications, especially in Fourier analysis. The material of this chapter lies at the foundation of the branch of mathematics called harmonic analysis.

In this chapter we will see that $L^2$ is a Hilbert space (we already really have all the bits of information we need to see this) and that in some sense the $L^2$-spaces (with different $\mu$'s) are the only Hilbert spaces. We will come to see how the problem that Fourier examined, about decomposing functions as infinite sums of other — somehow more basic — functions, is a problem best phrased and understood in the language of abstract Hilbert spaces. One of the triumphs of functional analysis is to take a very concrete problem — in this case Fourier decomposition — view it in an abstract setting, and use theoretical tools to obtain powerful results that can be translated back to the concrete setting. Fourier’s work certainly holds an important spot at the roots of functional analysis, and it motivated much early work in the development of the field.

Further Hilbert space theory appears in Section 5.4.

4.1 Orthonormal Sequences

During the second half of the eighteenth century and first decade of the nineteenth century, infinite sums of sines and cosines appeared as solutions to physical prob-
4.1 Orthonormal Sequences
lems then being studied. Daniel Bernoulli (1700–1782; Netherlands) suggested that these sums were solutions to the problem of modeling the vibrating string, and Joseph Fourier (1768–1830; France) proposed them as solutions to the problem of modeling heat flow. It is not really until the response to Fourier’s work that we see other mathematicians coming to grips with the challenge that these infinite sums truly posed: to understand the fundamental notions of convergence and continuity. Over the decades following the appearance of Fourier’s works on heat, the field of “real analysis” would be born in large part out of efforts to respond to the challenges that Fourier’s work raised in pure mathematics. Many of the great mathematicians of the period — perhaps most notably Cauchy, Riemann, and Weierstrass — did their most important work in the development of this field. For an excellent historical account of these mathematical developments, see [25]. Fourier begins with an arbitrary function $f$ on the interval from $-\pi$ to $\pi$ and states that if we can write

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx),$$

then it must be the case that the coefficients $a_k$ and $b_k$ are given by the formulas

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) \, dx, \quad k = 0, 1, 2, \ldots,$$

and

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) \, dx, \quad k = 1, 2, \ldots.$$

The big question is this: When is this decomposition actually possible? Even if the integrals involved make sense, does the series converge? If it does converge, what type of convergence (pointwise, uniform, etc.) do we get? Even if the series converges in some sense, does it converge to $f$?

The immediate goal is to show you how these questions about Fourier series can be treated in the abstract setting of an inner product space.

Let us now take stock of what we already know by gathering our information about $L^2$. First, recall that $L^2 = L^2(\mu)$, for any abstract measure space $(X, \mathcal{R}, \mu)$, denotes the collection of all measurable functions $f : X \to \mathbb{C}$ such that the integral

$$\int_X |f|^2 \, d\mu$$

$^1$Daniel Bernoulli is the nephew of James Bernoulli, who was mentioned at the beginning of Section 3.1. The Bernoulli family produced several distinguished mathematicians and physicists; at least twelve members of the family achieved distinction in at least one of these fields.