In this chapter you will read about the beginning material of operator theory. The chapter is written with the aim of getting to spectral theory as quickly as possible. Matrices are examples of linear operators. They transform one linear space into another and do so linearly. “Spectral values” are the infinite-dimensional analogues of eigenvalues in the finite-dimensional situation. Spectral values can be used to decompose operators, in much the same way that eigenvalues can be used to decompose matrices. You will see an example of this sort of decomposition in the last section of this chapter, where we prove the spectral theorem for compact Hermitian operators. One of the most important open problems in operator theory at the start of the twenty-first century is the “invariant subspace problem.” In the penultimate section of this chapter we give a description of this problem and discuss some partial solutions to it. We also let the invariant subspace problem serve as our motivation for learning a bit about operators on Hilbert space. The material found at the end of Section 3 (from Theorem 5.7 onwards) through the last section (Section 5) of the chapter is not usually covered in an undergraduate course. This material is sophisticated, and will probably seem more difficult than other topics we cover.

Further basic linear operator theory can be found in Section 6.3.

5.1 Basic Definitions and Examples

We start by considering two real (or complex) linear spaces, $X$ and $Y$. A mapping $T$ from $X$ to $Y$ is called a linear operator if the domain of $T$, $D_T$, is a linear
subspace of $X$, and if

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$$

for all $x, y \in D_T$ and all real (or complex) scalars $\alpha$ and $\beta$. Notice that any linear map satisfies $T(0) = 0$. In this context it is common to write $Tx$ in place of $T(x)$, and unless otherwise stated, $D_T$ is taken to be all of $X$. The first thing we want to do is establish a reasonable list of examples of linear operators.

**Example 1.** As should be familiar from linear algebra, any real $m \times n$ matrix $(a_{ij})$ defines a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^m$ via

$$
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n
\end{pmatrix}
= \begin{pmatrix}
  \sum_{j=1}^n a_{1j}x_j, \\
  \sum_{j=1}^n a_{2j}x_j, \\
  \vdots \\
  \sum_{j=1}^n a_{mj}x_j
\end{pmatrix}.$$

**Example 2.** An "infinite matrix" $(a_{ij})$, $i, j = 1, 2, \ldots$, can be used to represent an operator on a sequence space. For example, the infinite matrix

$$
\begin{pmatrix}
  0 & 1 & 0 & 0 & \cdots \\
  0 & 0 & 1 & 0 & \cdots \\
  0 & 0 & 0 & 1 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

represents the linear operator $T$ acting on $\ell^2 = \ell^2(\mathbb{N})$ (or $\ell^\infty$, etc.) given by

$$T(x_1, x_2, \ldots) = (x_2, x_3, \ldots).$$

This is an example of what is called a shift operator. More specifically, it is likely to be referred to as the "backward unilateral shift," or "unilateral shift," or "left shift." We will use the last of these names. Another important shift is the "right shift," defined by

$$S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots).$$

What is the (infinite) matrix representing this shift? There are also "weighted shifts." For example, the sequence of 1’s in the matrix of $T$ can be replaced by a suitable sequence $\{a_i\}_{i=1}^\infty$ of scalars, and the weighted shift thus constructed sends $(x_1, x_2, \ldots)$ to $(a_1x_2, a_2x_3, \ldots)$. The class of shift operators plays an important role in the theory of operators on Hilbert spaces.

**Example 3.** For our first example of a linear operator on a function space, we observe that the map

$$Tf = \int_0^1 f(t) dt$$

defines a linear operator $T : C([a, b]) \to \mathbb{R}$. 