In this chapter we study the diversity of the distribution of capital in an equity market. Heuristically speaking, a market is “diverse” if the capital is spread among a reasonably large number of stocks. We show that the excess growth rate of the market is related to the diversity of the capital distribution, and we use this relationship to study the long-term behavior of market diversity under the hypothesis that all the stocks have the same growth rate. It might seem that in such a market, diversity would naturally be maintained, but we shall see that this is not so, and in fact, such markets have a tendency to concentrate capital into single stocks. Dividend payments are a natural means to maintain market diversity, and we investigate the structure of this mechanism. Finally, we propose market entropy as a measure of market diversity, and study a derived portfolio called the entropy-weighted portfolio.

To analyze the long-term behavior of stocks, portfolios, or the market itself, it is appropriate that we consider the time-average values rather than the expected values of the processes under consideration. In practice, we are able to observe the time-average value, whereas the expected value is merely a theoretical construct. Hence, for the growth rate $\gamma_i$ of a stock $X_i$, we shall consider

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \gamma_i(t) \, dt$$

rather than $E\gamma_i(t)$. Likewise, for a market weight $\mu_i$, we shall study

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \log \mu_i(t) \, dt$$

rather than $E \log \mu_i(t)$.

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2. Stock Market Behavior

2.1 The Long-Term Behavior of the Market

In this section we shall investigate the long-term relative performance of the stocks in the market. This will also allow us to characterize the long-term behavior of certain simple portfolios. For some of the results here, we need to impose a structural condition on the market.

2.1.1 Definition. The market \( \mathcal{M} \) is coherent if for \( i = 1, \ldots, n \),

\[
\lim_{t \to \infty} \frac{1}{t} \log \mu_i(t) = 0, \quad \text{a.s.} \quad (2.1.1)
\]

Since \( \log \mu_i(t) < 0 \), condition (2.1.1) holds if none of the stocks declines too rapidly. Note that since \( \mu_i(t) = \frac{X_i(t)}{Z_{\mu}(t)} \), (2.1.1) is equivalent to

\[
\lim_{t \to \infty} t^{-1} (\log X_i(t) - \log Z_{\mu}(t)) = 0, \quad \text{a.s.} \quad (2.1.2)
\]

2.1.2 Proposition. Let \( \mathcal{M} \) denote the market with stocks \( X_1, \ldots, X_n \).

Then the following statements are equivalent:

(i) \( \mathcal{M} \) is coherent;

(ii) for \( i = 1, \ldots, n \),

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T (\gamma_i(t) - \gamma_{\mu}(t)) dt = 0, \quad \text{a.s.;}
\]

(iii) for \( i, j = 1, \ldots, n \),

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T (\gamma_i(t) - \gamma_j(t)) dt = 0, \quad \text{a.s.}
\]

Proof. We shall prove that (i) implies (ii) implies (iii) implies (i).

Suppose \( \mathcal{M} \) is coherent. Then (2.1.2) states that for \( i = 1, \ldots, n \),

\[
\lim_{T \to \infty} \frac{1}{T} (\log X_i(T) - \log Z_{\mu}(T)) = 0, \quad \text{a.s.}
\]

By Proposition 1.3.1,

\[
\lim_{T \to \infty} \frac{1}{T} \left( \log Z_{\mu}(T) - \int_0^T \gamma_{\mu}(t) dt \right) = 0, \quad \text{a.s.,}
\]

and by Corollary 1.3.3,

\[
\lim_{T \to \infty} \frac{1}{T} \left( \log X_i(T) - \int_0^T \gamma_i(t) dt \right) = 0, \quad \text{a.s.}
\]

These three equations imply condition (ii).

Condition (iii) follows immediately from condition (ii).

Now, suppose that condition (iii) holds. It is convenient here to explicitly show the dependence of all random variables and processes on \( \omega \in \Omega \).