A large area of current research interest is centered around the theory of operators on Hilbert space. Several other chapters in this book will be devoted to this topic.

There is a marked contrast here between Hilbert spaces and the Banach spaces that are studied in the next chapter. Essentially all of the information about the geometry of Hilbert space is contained in the preceding chapter. The geometry of Banach space lies in darkness and has attracted the attention of many talented research mathematicians. However, the theory of linear operators (linear transformations) on a Banach space has very few general results, whereas Hilbert space operators have an elegant and well-developed general theory. Indeed, the reason for this dichotomy is related to the opposite status of the geometric considerations. Questions concerning operators on Hilbert space don't necessitate or imply any geometric difficulties.

In addition to the fundamentals of operators, this chapter will also present an interesting application to differential equations in Section 6.

§1. Elementary Properties and Examples

The proof of the next proposition is similar to that of Proposition 1.3.1 and is left to the reader.

1.1. **Proposition.** Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $A: \mathcal{H} \to \mathcal{K}$ a linear transformation. The following statements are equivalent.

(a) $A$ is continuous.

(b) $A$ is continuous at $0$. 
(c) *A is continuous at some point.*
(d) *There is a constant \( c > 0 \) such that \( \|Ah\| \leq c\|h\| \) for all \( h \) in \( \mathcal{H} \).

As in (I.3.3), if
\[
\|A\| = \sup \{\|Ah\| : h \in \mathcal{H}, \|h\| \leq 1\},
\]
then
\[
\|A\| = \sup \{\|Ah\| : \|h\| = 1\}
= \sup \{\|Ah\|/\|h\| : h \neq 0\}
= \inf \{c > 0 : \|Ah\| \leq c\|h\|, \ h \in \mathcal{H}\}.
\]
Also, \( \|Ah\| \leq \|A\| \|h\| \). \( \|A\| \) is called the *norm* of \( A \) and a linear transformation with finite norm is called *bounded.* Let \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \) be the set of bounded linear transformations from \( \mathcal{H} \) into \( \mathcal{K} \). For \( \mathcal{H} = \mathcal{K}, \mathcal{B}(\mathcal{H}, \mathcal{K}) = \mathcal{B}(\mathcal{K}). \) Note that \( \mathcal{B}(\mathcal{H}, \mathcal{F}) \) = all the bounded linear functionals on \( \mathcal{H} \).

### 1.2. Proposition
(a) *If \( A \) and \( B \in \mathcal{B}(\mathcal{H}, \mathcal{K}), \) then \( A + B \in \mathcal{B}(\mathcal{H}, \mathcal{K}), \) and \( \|A + B\| \leq \|A\| + \|B\|.\)
(b) *If \( \alpha \in \mathcal{F} \) and \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}), \) then \( \alpha A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) and \( \|\alpha A\| = |\alpha| \|A\|.\)
(c) *If \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) and \( B \in \mathcal{B}(\mathcal{K}, \mathcal{L}), \) then \( BA \in \mathcal{B}(\mathcal{H}, \mathcal{L}) \) and \( \|BA\| \leq \|B\| \|A\|.\)

**Proof.** Only (c) will be proved; the rest of the proof is left to the reader. If \( k \in \mathcal{K}, \) then \( \|Bk\| \leq \|B\| \|k\|. \) Hence, if \( h \in \mathcal{H}, \ k = Ah \in \mathcal{K} \) and so \( \|BAh\| \leq \|B\| \|Ah\| \leq \|B\| \|A\| \|h\|\).

By virtue of the preceding proposition, \( d(A, B) = \|A - B\| \) defines a metric on \( \mathcal{B}(\mathcal{H}, \mathcal{K}). \) So it makes sense to consider \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \) as a metric space. This will not be examined closely until later in the book, but later in this chapter the idea of the convergence of a sequence of operators will be used.

### 1.3. Example
If \( \dim \mathcal{H} = n < \infty \) and \( \dim \mathcal{K} = m < \infty \), let \( \{e_1, \ldots, e_n\} \) be an orthonormal basis for \( \mathcal{H} \) and let \( \{e_1, \ldots, e_m\} \) be an orthonormal basis for \( \mathcal{K} \). It can be shown that every linear transformation from \( \mathcal{H} \) into \( \mathcal{K} \) is bounded (Exercise 3). If \( 1 \leq j \leq n, 1 \leq i \leq m, \) let \( \alpha_{ij} = \langle Ae_j, e_i \rangle. \) Then the \( m \times n \) matrix \( (\alpha_{ij}) \) represents \( A \) and every such matrix represents an element of \( \mathcal{B}(\mathcal{H}, \mathcal{K}). \)

### 1.4. Example
Let \( l^2 \equiv l^2(\mathbb{N}) \) and let \( e_1, e_2, \ldots \) be its usual basis. If \( A \in \mathcal{B}(l^2), \) form \( \alpha_{ij} = \langle Ae_j, e_i \rangle. \) The infinite matrix \( (\alpha_{ij}) \) represents \( A \) as finite matrices represent operators on finite dimensional spaces. However, this representation has limited value unless the matrix has a special form. One difficulty is that it is unknown how to find the norm of \( A \) in terms of