Axiomatic Systems and Finite Geometries

1.1. Gaining Perspective

Finite geometries were developed in the late 19th century, in part to demonstrate and test the axiomatic properties of "completeness," "consistency," and "independence." They are introduced in this chapter to fulfill this historical role and to develop both an appreciation for and an understanding of the revolution in mathematical and philosophical thought brought about by the development of non-Euclidean geometry. In addition, finite geometries provide relatively simple axiomatic systems in which we can begin to develop the skills and techniques of geometric reasoning. The finite geometries introduced in Sections 1.3 and 1.5 also illustrate some of the fundamental properties of non-Euclidean and projective geometry.

Even though finite geometries were developed as abstract systems, mathematicians have applied these abstract ideas in designing statistical experiments using Latin squares and in developing error-correcting codes in computer science. Section 1.4 develops a simple error-correcting code and shows its connection with finite projective geometries. The application of finite affine geometries to the building of Latin squares is equally intriguing. Since Latin squares are clearly described in several readily accessible sources, the reader is encouraged to explore this topic by consulting the resources listed at the end of this chapter.

1.2. Axiomatic Systems

The study of any mathematics requires an understanding of the nature of deductive reasoning, and geometry has been singled out for introducing this methodology to secondary-school students. There are important historical reasons for choosing geometry to fulfill this role, but these reasons are seldom revealed to secondary-school initiates. This section introduces the terminology essential for a discussion of deductive reasoning so that the extraordi-
nary influence of the history of geometry on the modern understanding of deductive systems will become evident.

Deductive reasoning takes place in the context of an organized logical structure called an axiomatic (or deductive) system. Such a system consists of the following components:

1. Undefined terms.
2. Defined terms.
3. Axioms.
4. A system of logic.
5. Theorems.

Undefined terms are included since it is not possible to define all terms without resorting to circular definitions. In geometrical systems these undefined terms frequently, but not necessarily, include "point," "line," "plane," and "on." Defined terms are not actually necessary, but in nearly every axiomatic system certain phrases involving undefined terms are used repeatedly. Thus it is more efficient to substitute a new term, that is, a defined term, for each of these phrases whenever they occur. For example, in Euclidean geometry we substitute the term "parallel lines" for the phrase "lines which do not intersect." Furthermore, it is impossible to prove all statements constructed from the defined and undefined terms of the system without circular reasoning, just as it is impossible to define all terms. So an initial set of statements is accepted without proof. The statements that are accepted without proof are known as axioms. From the axioms, other statements can be deduced or proved using the rules of inference of a system of logic (usually Aristotelian). These latter statements are called theorems.

As noted earlier, the axioms of a system must be statements constructed using the terms of the system. But they cannot be arbitrarily constructed since an axiom system must be consistent.

**Definition 1.1.** An axiomatic system is said to be consistent if there do not exist in the system any two axioms, any axiom and theorem, or any two theorems that contradict each other.

It should be clear that it is essential that an axiomatic system be consistent since a system in which both a statement and its negation can be proved is worthless. However, it soon becomes evident that it would be difficult to verify consistency directly from this definition since all possible theorems would have to be considered. Instead, models are used for establishing consistency. A model of an axiomatic system is obtained by assigning interpretations to the undefined terms so as to convert the axioms into true statements in the interpretations. If the model is obtained by using interpretations that are objects and relations adapted from the real world, we say we have established absolute consistency. In this case, statements corresponding to any contradictory theorems would lead to contradictory statements in the model, but