5

Higher-Order Linear Differential Equations

5.0 Introduction

Linear differential equations of higher order have useful and interesting applications, just as first-order differential equations do. We study linear differential equations of higher order in this chapter. The word linear in the chapter title should suggest that techniques for solving linear equations will be important. What is somewhat unexpected is that we have to appeal to the theory of solving polynomial equations in one variable. Though the solution technique for first-order equations gave us a complete solution in essentially one step, this is not the case here. For the first time we have to solve the homogeneous and nonhomogeneous equations separately and by different methods.

When solving the homogeneous equation, we need to find the roots of polynomials. The notion of linearly independent solutions becomes centrally important for the first time. The Wronskian determinant is introduced to test for linear independence. It is the determinant of the Wronskian matrix upon which the technique of variation of parameters is based. Variation of parameters and the method of undetermined coefficients are used to find a particular solution to nonhomogeneous problems, once the homogeneous problem has been completely solved.

This chapter leans heavily on linear algebra for its theory. You are shown how to use Mathematica to perform many of the steps that are simple in theory but very hard in practice. The idea behind the use of Mathematica is to free you from computational burdens so that you can concentrate on the meaning of our activities, rather than on the many details that arise as we seek solutions.

In this chapter we once again see all of the elementary functions that you have studied polynomials, the natural exponential, the sine and cosine functions, and the hyperbolic functions. The theory of linear differential equations with constant coefficients is built on these functions. The natural logarithm plays an important role in Chapter 8 where differential equations with variable coefficients are studied. Matrices become important to us in this chapter, and remain so throughout the rest of the text.
The Mathematica function NDSolve can be used to obtain accurate numerical solutions to the differential equations studied in this chapter, but this accuracy extends only to the values of the solution function, not the derivatives of the solution. This is because NDSolve approximates the solution by a cubic spline, a sequence of cubic polynomials that are defined over short intervals and adjacent polynomials agree in function value and slope at the endpoint that is common to their intervals of definition. This serves very well for accurately determining points along the solution curve, but these cubics do not accurately convey slope information.

If you need slope information, then you need techniques from Chapt. 9 for expressing a higher order differential equation as a system of equations that explicitly define the derivatives you need. In that context, NDSolve does give accurate values for the derivatives that the differential system explicitly defines. Be alert to what you are asking a numerical method to do, to be sure that it is capable of providing what you seek.

5.1 The Fundamental Theorem

In order to see why we can expect to find solutions for differential equations, we state the fundamental theorem, which says that there are solutions and gives conditions under which there is only one solution. Of course, knowing that there are solutions is not at all the same thing as being able to actually find a solution. But there is a large class of important differential equations, the linear differential equations with constant coefficients, where we can actually write down a formula for the solutions. But even in cases such as these, when it is clear what has to be done to obtain a solution, the theory is much easier than the practice. We will find that Mathematica eases the computational burden immensely, but that there are places where the theory says something exists and another piece of theory says that we cannot necessarily write out an actual solution. In these cases we can often get approximate solutions that are very close to the theoretically exact solutions we seek. But when we are calculating with approximate objects an immediate question is: just how good is the approximation? Questions such as these are covered in courses in numerical analysis.

The Fundamental Theorem

Theorem 5.1 (Existence and Uniqueness). Given an open subset $U$ of $(n+1)$-space $\mathbb{R}^{n+1}$ and a point $P = (x_0, a_0, a_1, a_2, \ldots, a_{n-1})$ of $U$. Suppose that the real-valued function $f$ is defined and continuous on $U$ and there is a positive number $M$ so that if $(x, u_1, u_2, \ldots, u_n)$ and $(x, v_1, v_2, \ldots, v_n)$ are in $U$ it follows that

$$|f(x, u_1, u_2, \ldots, u_n) - f(x, v_1, v_2, \ldots, v_n)| \leq M(|u_1 - v_1| + |u_2 - v_2| + \cdots + |u_n - v_n|).$$

Then there is exactly one solution of the $n$th-order initial value problem