CHAPTER VI
Limits of Functions

For real-valued functions of a real variable, the student already knows what it means for a sequence \( f_1, f_2, \ldots \) of functions to tend uniformly, or to tend simply, to a function \( f \). In this chapter we study these concepts in the general setting of metric spaces. We obtain in this way certain of the 'infinite-dimensional' spaces alluded to in the Introduction, and, thanks to Ascoli's theorem, the compact subsets of these spaces.

6.1. Uniform Convergence

6.1.1. Let \( X \) and \( Y \) be two sets. The mappings of \( X \) into \( Y \) form a set which will henceforth be denoted \( \mathcal{F}(X, Y) \).

6.1.2. Let \( X \) be a set, \( Y \) a metric space. For \( f, g \in \mathcal{F}(X, Y) \), we set
\[
d(f, g) = \sup_{x \in X} d(f(x), g(x)) \in [0, +\infty].
\]

Let us show that \( d \) is a metric (with possibly infinite values) on \( \mathcal{F}(X, Y) \). If \( d(f, g) = 0 \) then, for every \( x \in X \),
\[
d(f(x), g(x)) = 0,
\]
therefore \( f(x) = g(x) \); thus \( f = g \). It is clear that \( d(f, g) = d(g, f) \). Finally, if \( h \in \mathcal{F}(X, Y) \) then, for every \( x \in X \),
\[
d(f(x), h(x)) \leq d(f(x), g(x)) + d(g(x), h(x)) \leq d(f, g) + d(g, h);
\]
this being true for all \( x \in X \), we infer that
\[
d(f, h) \leq d(f, g) + d(g, h).
\]
6.1. Uniform Convergence

This metric is called the **metric of uniform convergence** on \( \mathcal{F}(X, Y) \). The corresponding topology is called the **topology of uniform convergence**.

6.1.3. Let \( f, f_1, f_2, f_3, \ldots \in \mathcal{F}(X, Y) \). To say that \( (f_n) \) tends to \( f \) for this topology means that \( \sup_{x \in X} d(f(x), f_n(x)) \to 0 \), in other words: for every \( \varepsilon > 0 \) there exists an \( N \) such that

\[
 n \geq N \Rightarrow d(f_n(x), f(x)) \leq \varepsilon \quad \text{for all } x \in X. 
\]

We then also say that the sequence \( (f_n) \) **tends uniformly to** \( f \).

6.1.4. Let \( \Lambda \) be a set equipped with a filter base \( \mathcal{B} \). For every \( \lambda \in \Lambda \), let \( f_\lambda \in \mathcal{F}(X, Y) \). Let \( f \in \mathcal{F}(X, Y) \). To say that \( f_\lambda \) tends to \( f \) along \( \mathcal{B} \) for the topology of uniform convergence means: for every \( \varepsilon > 0 \), there exists \( B \in \mathcal{B} \) such that

\[
 \lambda \in B \Rightarrow d(f_\lambda(x), f(x)) \leq \varepsilon \quad \text{for all } x \in X. 
\]

We then also say that \( f_\lambda \) tends to \( f \) uniformly along \( \mathcal{B} \).

6.1.5. **Example.** Take \( X = Y = \Lambda = \mathbb{R} \). For filter base \( \mathcal{B} \) on \( \Lambda \), take the set of intervals \( (a, +\infty) \). For \( \lambda \in \mathbb{R} \) and \( x \in \mathbb{R} \), set \( f_\lambda(x) = e^{-\lambda(x^2 + 1)} \). Then \( f_\lambda \) tends to 0 uniformly as \( \lambda \to +\infty \) (that is, along \( \mathcal{B} \)). For, let \( \varepsilon > 0 \). There exists \( a \in \mathbb{R} \) such that \( \lambda \geq a \Rightarrow e^{-\lambda} \leq \varepsilon \). Then (provided \( a \geq 0 \)):

\[
 \lambda \geq a \Rightarrow |e^{-\lambda(x^2 + 1)} - 0| = e^{-\lambda(x^2 + 1)} \leq e^{-\lambda} \leq \varepsilon \quad \text{for all } x \in \mathbb{R}. 
\]

6.1.6. **Theorem.** Let \( X \) be a set, \( Y \) a complete metric space. Then the metric space \( \mathcal{F}(X, Y) \) is complete.

Let \( (f_n) \) be a Cauchy sequence in \( \mathcal{F}(X, Y) \). Let \( x \in X \). Then

\[
 d(f_m(x), f_n(x)) \leq d(f_m, f_n) \to 0 \quad \text{as } m, n \to \infty, 
\]

thus \( (f_n(x)) \) is a Cauchy sequence in \( Y \), consequently has a limit in \( Y \) which we denote \( f(x) \). We have thus defined a mapping \( f \) of \( X \) into \( Y \).

Let \( \varepsilon > 0 \). There exists an \( N \) such that

\[
 m, n \geq N \Rightarrow d(f_m, f_n) \leq \varepsilon 
\]

\[
 \Rightarrow d(f_m(x), f_n(x)) \leq \varepsilon \quad \text{for all } x \in X. 
\]

We provisionally fix \( x \in X \) and \( m \geq N \). As \( n \to \infty \), the preceding inequality yields in the limit

\[
 d(f_m(x), f(x)) \leq \varepsilon. 
\]

This being true for all \( x \in X \), we have \( d(f_m, f) \leq \varepsilon \). Thus,

\[
 m \geq N \Rightarrow d(f_m, f) \leq \varepsilon. 
\]

In other words, \( (f_m) \) tends to \( f \) in \( \mathcal{F}(X, Y) \).