Chapter 5

THE NATURAL NUMBERS

Our fundamental intuitive understanding of the natural numbers is that there is a (least) number 0, that every number \( n \) has a successor \( S_n \), and that if we start with 0 and construct in sequence the successor of every number
\[
0, \; S0 = 1, \; S1 = 2, \; S2 = 3, \ldots
\]
forever, then in time we will reach every natural number. In set theoretic terms we can capture this intuition by the following axiomatic characterization.

5.1. Definition. A **system of natural numbers** is any structured set
\[
(N, 0, S) = (N, (0, S))
\]
which satisfies the following conditions.

1. \( N \) is a set which contains the element \( 0, \; 0 \in N \).
2. \( S \) is a function on \( N \), \( S : N \to N \).
3. \( S \) is an injection, \( S_n = S_m \implies n = m \).
4. For each \( n \in N \), \( S_n \neq 0 \).
5. **Induction Principle.** For each \( X \subseteq N \),
\[
[0 \in X \land (\forall n \in N)[n \in X \implies S_n \in X]] \implies X = N.
\]

These obvious properties of the natural numbers are called the **axioms of Peano** in honor of the Italian logician and mathematician who first proposed them as an axiomatic foundation of number theory. Most significant among them is the Induction Principle, whose typical application is illustrated in the proof of the next lemma.

5.2. Lemma. In a system of natural numbers \( (N, 0, S) \), every element \( n \neq 0 \) is a successor,
\[
n \neq 0 \implies (\exists m \in N)[n = Sm],
\]
and for each \( n \), \( S_n \neq n \).

**Proof.** To prove the first assertion by the Induction Principle, it is enough to show that the set

\[
X = \{ n \in N \mid n = 0 \lor (\exists m \in N)[n = Sm]\}
\]

satisfies the conditions

\[
0 \in X, \quad (\forall n \in N)[n \in X \implies S_n \in X],
\]

and both of these are obvious from the definition of \( X \). In the same way, for the second assertion it is enough to verify that \( S0 \neq 0 \) (which holds because, in general, \( S_n \neq 0 \)) and that \( S_n \neq n \implies SSn \neq Sn \): this holds because \( S \) is one-to-one, so that \( SSn = Sn \implies Sn = n \).

Number theory is one of the richest and most sophisticated fields of mathematics and it is by no means obvious that it can be developed on the basis of these five, simple properties; in fact, they do not suffice, one also needs to use set theory which (in its naive form) Peano took for granted, as part of "logic." Here we will only show that the axioms imply the first, most basic properties of addition, multiplication and the ordering on the natural numbers, which is all we need. The proofs we will give, however, are characteristic samples of the use of the Peano axioms in the more advanced parts of the theory of numbers.

If number theory can be developed from the Peano axioms, then to give a faithful representation of the natural numbers in set theory, it is enough to prove from the axioms the following two theorems.

5.3. **Existence Theorem for the Natural Numbers.** There exists at least one system of natural numbers \((N, 0, S)\).

5.4. **Uniqueness Theorem for the Natural Numbers.** For any two systems of natural numbers \((N_1, 0_1, S_1)\) and \((N_2, 0_2, S_2)\), there exists one (and only one) bijection

\[
\pi : N_1 \leftrightarrow N_2,
\]

which satisfies the identities

\[
\pi(0_1) = 0_2, \quad \pi(S_1n) = S_2\pi(n) \quad (n \in N_1).
\]

A bijection \( \pi \) which satisfies these identities is an **isomorphism** of the two systems \((N_1, 0_1, S_1)\) and \((N_2, 0_2, S_2)\), so that the theorem asserts that any two systems of natural numbers are isomorphic.

The Existence Theorem is very simple and we can prove it immediately.