6

The Supercritical Phase

6.1 Introduction

In this chapter we shall consider bond percolation on \( \mathbb{Z}^d \) where \( d \geq 2 \), and we shall suppose that \( p > p_c \). In this case, there exists at least one infinite open cluster, and the first main result of the chapter is that there exists a unique such cluster.

In the subcritical phase we were interested in such quantities as \( P_p(|C(0)| \geq n) \) and \( P_p(0 \leftrightarrow x) \), and particularly in their asymptotic behaviour for large \( n \) and \( |x| \). Such quantities are of less interest in the supercritical phase, since their asymptotic behaviour is dominated to the first order by the probabilities that the origin is in the infinite open cluster, or that both the origin and \( x \) are in this cluster, respectively. Of greater interest are the corresponding probabilities defined in terms of finite open clusters. That is to say, we shall study the asymptotic behaviour of \( P_p(n \leq |C(0)| < \infty) \) and \( P_p(0 \leftrightarrow x, |C(0)| < \infty) \).

Here is a guide to the chapter. We begin by proving that, if there exists an infinite open cluster, then there is exactly one such cluster. As a consequence of this, the percolation probability \( \theta(p) \) is a continuous function of \( p \) on \((p_c, 1]\). There follow later two sections devoted respectively to the exponential decay of (a) the tail of the radius of a finite open cluster, and (b) the truncated connectivity function \( \tau_f^L(0, x) = P_p(0 \leftrightarrow x, |C| < \infty) \) when \( x \) is distant from the origin; we prove that these quantities decay strictly exponentially (that is, with strictly positive rate) at least when \( p \) exceeds \( p_c^+ \), the critical probability of the half-space \( \{ x \in \mathbb{Z}^d : x_1 \geq 0 \} \). This is followed by a proof that the cluster size distribution decays slower than exponentially in the supercritical phase. In Section 6.8 we prove that the percolation probability \( \theta(p) \), the mean size \( \chi^f(p) \) of the finite open cluster at the origin, and the number \( k(p) \) of open clusters per vertex are infinitely differentiable functions of \( p \) when \( p > p_c^+ \). It is con-
jectured that $p_c^+ = p_c$, and it is a major open problem to prove or disprove this. Finally, we reach some simple conclusions about the geometry of the infinite open cluster.

### 6.2 Uniqueness of the Infinite Open Cluster

The principal result of this section is the following: for any value of $p$ for which there is positive probability of an infinite open cluster, there exists almost surely exactly one infinite open cluster.

**6.1 Theorem. Uniqueness of the infinite open cluster.** If $p$ is such that $\theta(p) > 0$ then

\[
P_p(\text{there is exactly one infinite open cluster}) = 1.
\]

In proving the uniqueness of the infinite open cluster, we shall make use of the following lemma. Let $I_{uv}$ be the event that the vertices $u$ and $v$ are in different infinite open clusters of $\mathbb{L}^d$.

**6.2 Lemma.** Suppose that $\theta(p) > 0$, and let $x$ and $y$ be neighbours in $\mathbb{L}^d$. If

\[
P_p(I_{xy}) = 0,
\]

then

\[
P_p(\text{there is a unique infinite open cluster}) = 1.
\]

Clearly

\[
P_p(\text{there are two or more infinite open clusters}) \leq \sum_{u, v \in \mathbb{Z}^d} P_p(I_{uv}),
\]

so that the infinite open cluster is almost surely unique whenever $P_p(I_{uv}) = 0$ for all pairs $u, v$ of distinct vertices; by the lemma, it is sufficient to check this condition for a pair $u, v$ of neighbours.

**Proof of Lemma (6.2).** Of the several ways of proving this, the following is possibly the simplest. It is trivial that the infinite open cluster is unique almost surely if $p = 1$, and so we assume that $0 < p < 1$. Let $x$ and $y$ be two neighbours of $\mathbb{L}^d$. For any box $B$ containing $x$ and $y$ in its interior, and any pair $u, v$ of distinct vertices in the surface $\partial B$ of $B$, we denote by $I_{uv}(B)$ and $J_{uvxy}(B)$ the two events illustrated in Figure 6.1. More precisely, $I_{uv}(B)$ is the event that $u$ and $v$ are in different infinite open clusters of the graph obtained from $\mathbb{L}^d$ by declaring every edge in $B$ to be closed, and $J_{uvxy}(B)$ is the event that every edge of $B$ is closed.