

## Ramanujan, Modular Equations, and Approximations to Pi or How to Compute One Billion Digits of Pi

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**Preface.** The year 1987 was the centenary of Ramanujan's birth. He died in 1920. Had he not died so young, his presence in modern mathematics might be more immediately felt. Had he lived to have access to powerful algebraic manipulation software, such as MACSYMA, who knows how much more spectacular his already astonishing career might have been.

This article will follow up one small thread of Ramanujan's work which has found a modern computational context, namely, one of his approaches to approximating pi. Our experience has been that as we have come to understand these pieces of Ramanujan's work, as they have become mathematically demystified, and as we have come to realize the intrinsic complexity of these results, we have come to realize how truly singular his abilities were. This article attempts to present a considerable amount of material and, of necessity, little is presented in detail. We have, however, given much more detail than Ramanujan provided. Our intention is that the circle of ideas will become apparent and that the finer points may be pursued through the indicated references.

**1. Introduction.** There is a close and beautiful connection between the transformation theory for elliptic integrals and the very rapid approximation of pi. This connection was first made explicit by Ramanujan in his 1914 paper "Modular Equations and Approximations to  $\pi$ " [26]. We might emphasize that Algorithms 1 and 2 are not to be found in Ramanujan's work, indeed no recursive approximation of  $\pi$  is considered, but as we shall see they are intimately related to his analysis. Three central examples are:

**Sum 1. (Ramanujan)**

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{[1103 + 26390n]}{396^{4n}}.$$

**Algorithm 1.** Let  $\alpha_0 := 6 - 4\sqrt{2}$  and  $y_0 := \sqrt{2} - 1$ .  
Let

$$y_{n+1} := \frac{1 - (1 - y_n^4)^{1/4}}{1 + (1 - y_n^4)^{1/4}}$$

and

$$\alpha_{n+1} := (1 + y_{n+1})^4 \alpha_n - 2^{2n+3} y_{n+1} (1 + y_{n+1} + y_{n+1}^2).$$

Then

$$0 < \alpha_n - 1/\pi < 16 \cdot 4^n e^{-2 \cdot 4^n \pi}$$

and  $\alpha_n$  converges to  $1/\pi$  *quartically* (that is, with order four).

**Algorithm 2.** Let  $s_0 := 5(\sqrt{5} - 2)$  and  $\alpha_0 := 1/2$ .

Let

$$s_{n+1} := \frac{25}{(z + x/z + 1)^2 s_n},$$

where

$$x := 5/s_n - 1 \quad y := (x - 1)^2 + 7$$

and

$$z := \left[ \frac{1}{2} x (y + \sqrt{y^2 - 4x^3}) \right]^{1/5}.$$

Let

$$\alpha_{n+1} := s_n^2 \alpha_n - 5^n \left\{ \frac{s_n^2 - 5}{2} + \sqrt{s_n(s_n^2 - 2s_n + 5)} \right\}.$$

Then

$$0 < \alpha_n - \frac{1}{\pi} < 16 \cdot 5^n e^{-5^n \pi}$$

and  $\alpha_n$  converges to  $1/\pi$  *quintically* (that is, with order five).

Each additional term in Sum 1 adds roughly eight digits, each additional iteration of Algorithm 1 quadruples the number of correct digits, while each additional iteration of Algorithm 2 quintuples the number of correct digits. Thus a mere thirteen iterations of Algorithm 2 provide in excess of one billion decimal digits of  $\pi$ . In general, for us,  $p$ th-order convergence of a sequence  $\{\alpha_n\}$  to  $\alpha$  means that  $\alpha_n$  tends to  $\alpha$  and that

$$|\alpha_{n+1} - \alpha| \leq C |\alpha_n - \alpha|^p$$

for some constant  $C > 0$ . Algorithm 1 is arguably the most efficient algorithm currently known for the extended precision calculation of  $\pi$ . While the rates of convergence are impressive, it is the subtle and thoroughly nontransparent nature of these results and the beauty of the underlying mathematics that intrigue us most.

Watson [37], commenting on certain formulae of Ramanujan, talks of

*a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuovo of the Capella Medici and see before me the austere beauty of the four statues representing "Day," "Night," "Evening," and "Dawn" which Michelangelo has set over the tomb of Giuliano de' Medici and Lorenzo de' Medici.*

Sum 1 is directly due to Ramanujan and appears in [26]. It rests on a modular identity of order 58 and, like much of Ramanujan's work, appears without proof and with only scanty motivation. The first complete derivation we know of appears