Reprinted from the AMERICAN MATHEMATICAL MONTHLY
Vol. 96, No. 8, October 1989

Pi, Euler Numbers, and Asymptotic Expansions

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1. Introduction. Gregory's series for π, truncated at 500,000 terms, gives to forty places

$$4\sum_{k=1}^{500,000}\frac{(-1)^{k-1}}{2k-1} = 3.1415\underline{9}06535897932\underline{4}0462643383\underline{2}69502884197.$$

The number on the right is not π to forty places. As one would expect, the 6th digit after the decimal point is wrong. The surprise is that the next 10 digits are correct. In fact, only the 4 underlined digits aren't correct. This intriguing observation was sent to us by R. D. North [10] of Colorado Springs with a request for an explanation. The point of this article is to provide that explanation. Two related

[1]Research of the authors supported in part by NSERC of Canada.

L. Berggren et al., *Pi: A Source Book*
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examples, to fifty digits, are

$$\frac{\pi}{2} \doteq 2 \sum_{k=1}^{50,000} \frac{(-1)^{k-1}}{2k-1}$$

$$= 1.5707\underline{8}6326794897\underline{6}1923132\underline{1}1916397\underline{52}052098583\underline{3}1473\underline{88}$$

$$\begin{array}{cccc} 1 & -1 & 5 & -61 \end{array}$$

and

$$\log 2 \doteq \sum_{k=1}^{50,000} \frac{(-1)^{k+1}}{k}$$

$$= .693\underline{1}37\underline{1}80\underline{6}5994530\underline{9}39\underline{7}2321214\underline{7}41765680\underline{48}30013\underline{446572},$$

$$\begin{array}{ccccc} 1 & -1 & 2 & -16 & 272 \end{array}$$

where all but the underlined digits are correct. The numbers under the underlined digits are the numbers that must be added to correct these. The numbers 1, -1, 5, -61 are the first four *Euler numbers* while 1, -1, 2, -16, 272 are the first five *tangent numbers*. Our process of discovery consisted of generating these sequences and then identifying them with the aid of Sloane's *Handbook of Integer Sequences* [11]. What one is observing, in each case, is an asymptotic expansion of the error in Euler summation. The amusing detail is that the coefficients of the expansion are integers. All of this is explained by Theorem 1.

The standard facts we need about the *Euler numbers* $\{E_i\}$, the *tangent numbers* $\{T_i\}$, and the *Bernoulli numbers* $\{B_i\}$, may all be found in [1] or in [6]. The numbers are defined as the coefficients of the power series

$$\sec z = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n} z^{2n}}{(2n)!}, \tag{1.1}$$

$$\tan z = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{T_{2n+1} z^{2n+1}}{(2n+1)!} \qquad \text{and } T_0 = 1, \tag{1.2}$$

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n z^n}{n!}. \tag{1.3}$$

They satisfy the relations

$$\sum_{k=0}^{n} \binom{2n}{2k} E_{2k} = 0, \qquad E_{2n+1} = 0, \tag{1.4}$$

$$B_n = \frac{-nT_{n-1}}{2^n(2^n - 1)} \qquad n \geqslant 1, \tag{1.5}$$

and

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0. \tag{1.6}$$

These three identities allow for the easy generation of $\{E_n\}$, $\{T_n\}$, and $\{B_n\}$. The