2.1. Introduction

In evaluating the consequences of possible actions, two major problems are encountered. The first is that the values of the consequences may not have any obvious scale of measurement. For example, prestige, customer goodwill, and reputation are important to many businesses, but it is not clear how to evaluate their importance in a concrete way. A typical problem of this nature arises when a relatively exclusive company is considering marketing its "name" product in discount stores. The immediate profit which would accrue from increased sales is relatively easy to estimate, but the longterm effect of a decrease in prestige is much harder to deal with.

Even when there is a clear scale (usually monetary) by which consequences can be evaluated, the scale may not reflect true "value" to the decision maker. As an example, consider the value to you of money. Assume you have the opportunity to do a rather unpleasant task for $100. At your present income level, you might well value the 100 dollars enough to do the task. If, on the other hand, you first received a million dollars, the value to you of an additional $100 would be much less, and you would probably choose not to do the task. In other words, the value of $1,000,100 is probably not the same as the value of $1,000,000 plus the value of $100. As another example, suppose you are offered a choice between receiving a gift of $10,000 or participating (for free) in a gamble wherein you have a 50-50 chance of winning $0 or $25,000. Most of us would probably choose the sure $10,000. If this is the case, then the expected "value" of the gamble is less than $10,000. In the ensuing sections, a method of determining true value will be discussed. This will then be related to the development of the loss function.
2.2. Utility Theory

To work mathematically with ideas of "value," it will be necessary to assign numbers indicating how much something is valued. Such numbers are called utilities, and utility theory deals with the development of such numbers.

To begin, it is necessary to clearly delineate the possible consequences which are being considered. The set of all consequences of interest will be called the set of rewards, and will be denoted by \( R \). Quite frequently \( R \) will be the real line (such as when the consequences can be given in monetary terms), but often the elements of \( R \) will consist of nonnumerical quantities such as mentioned in the introduction.

Often there is uncertainty as to which of the possible consequences will actually occur. Thus the results of actions are frequently probability distributions on \( R \). Let \( P \) denote the set of all such probability distributions. It is usually necessary to work with values and preferences concerning probability distributions in \( P \). This would be easy to do if a real-valued function \( U(r) \) could be constructed such that the "value" of a probability distribution \( P \in P \) would be given by the expected utility \( E^P[U(r)] \). If such a function exists, it is called a utility function.

A precise formulation of the problem begins with the assumption that it is possible to state preferences among elements of \( P \). (If one cannot decide the relative worth of various consequences, there is no hope in trying to construct measures of their value.) The following notation will be used to indicate preferences.

**Definition 1.** If \( P_1 \) and \( P_2 \) are in \( P \), then \( P_1 < P_2 \) means that \( P_2 \) is preferred to \( P_1 \); \( P_1 = P_2 \) means that \( P_1 \) is equivalent to \( P_2 \); and \( P_1 \preceq P_2 \) means that \( P_1 \) is not preferred to \( P_2 \).

A reward \( r \in R \) will be identified with the probability distribution in \( P \) which gives probability one to the point \( r \). This probability distribution will be denoted \( (r) \). (See Section 1.4 for a similar use of this notational device.) Hence the above definition applies also to rewards.

The goal is to find a function \( U(r) \) which represents (through expected value) the true preference pattern on \( P \) of the decision maker. In other words, a function \( U \) is sought such that if \( P_1 \) and \( P_2 \) are in \( P \), then \( P_2 \) is preferred to \( P_1 \) if and only if

\[
E^P[U(r)] < E^P_2[U(r)].
\]

The function \( U \) is then the desired quantification of the decision maker's preference or value pattern, and can be called a utility function.

It is by no means clear that a utility function need exist. We will shortly give a brief discussion of certain conditions which guarantee the existence of a utility function. First, however, a useful method for constructing a utility function (assuming one exists) is given. In the construction, we will